Math 2280 - Final Exam

University of Utah

Fall 2013

Name: Solutions by Dylan Zwick

This is a 2 hour exam. Please show all your work, as a worked problem is required for full points, and partial credit may be rewarded for some work in the right direction.

Things You Might Want to Know

Definitions

$$\mathcal{L}(f(t)) = \int_0^\infty e^{-st} f(t)dt.$$

$$f(t) * g(t) = \int_0^t f(\tau)g(t-\tau)d\tau.$$

Laplace Transforms

$$\mathcal{L}(t^n) = \frac{n!}{s^{n+1}}$$

$$\mathcal{L}(e^{at}) = \frac{1}{s-a}$$

$$\mathcal{L}(\sin(kt)) = \frac{k}{s^2+k^2}$$

$$\mathcal{L}(\cos(kt)) = \frac{s}{s^2+k^2}$$

$$\mathcal{L}(\delta(t-a)) = e^{-as}$$

$$\mathcal{L}(u(t-a)f(t-a)) = e^{-as}F(s).$$

Translation Formula

$$\mathcal{L}(e^{at}f(t)) = F(s-a).$$

Derivative Formula

$$\mathcal{L}(x^{(n)}) = s^n X(s) - s^{n-1}x(0) - s^{n-2}x'(0) - \cdots - sx^{(n-2)}(0) - x^{(n-1)}(0).$$

Fourier Series Definition

For a function $f(t)$ of period $2L$ the Fourier series is:

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi t}{L}\right) + b_n \sin\left(\frac{n\pi t}{L}\right) \right).$$

$$a_n = \frac{1}{L} \int_{-L}^{L} f(t) \cos\left(\frac{n\pi t}{L}\right) dt$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(t) \sin\left(\frac{n\pi t}{L}\right) dt.$$

1. **Basic Definitions** (5 points)

   (a) (3 points) State the order of the differential equation

   $$e^x y^{(3)} - 2\sin(y) + 3x^4 y' = \ln x.$$

   Solution - 3rd Order

   (b) (2 points) Is the differential equation

   $$(x+1)y^{(2)} + 2y = 0$$

   linear or nonlinear?

   Solution - Nonlinear

2. **Undetermined Coefficients** (5 points)

   Use the method of undetermined coefficients to state the form of the particular solution to the differential equation

   $$y^{(3)} - y'' - 4y' + 4y = x^2 e^{2x} \sin(3x).$$

   You do not have to solve for the coefficients, or solve the differential equation.

   Solution - $(Ax^2 + Bx + C)e^{2x}\sin(3x) + (Dx^2 + Ex + F)e^{2x}\cos(3x)$.

3. **Converting to a First-Order System** (5 points)

   Convert the differential equation

$$y^{(3)} - y'' - 4y' + 4y = x^2e^{2x}\sin(3x).$$

   into an equivalent system of first-order equations.

   Solution -

$$\begin{aligned} y_1' &= y_2, \\ y_2' &= y_3, \\ y_3' &= y_3 + 4y_2 - 4y_1 + x^2e^{2x}\sin(3x). \end{aligned}$$

4. **Linear ODEs with Constant Coefficients** (5 points)

   Find the general solution to the homogeneous equation corresponding to the differential equation

$$y^{(3)} - y'' - 4y' + 4y = x^2e^{2x}\sin(3x).$$

   Hint - One root of the polynomial $r^3 - r^2 - 4r + 4$ is $r = 2$.

   Solution - The corresponding homogeneous equation is

$$y^{(3)} - y'' - 4y' + 4y = 0.$$

   The characteristic polynomial for this ODE is $r^3 - r^2 - 4r + 4 = (r-1)(r+2)(r-2)$. This polynomial has roots $r = 1, \pm 2$. So, the corresponding general solution is:

$$y(x) = c_1e^x + c_2e^{2x} + c_3e^{-2x}.$$

5. **Solving Systems of Linear ODEs** (15 points)

Find the general solution to the system of ODEs

$$\mathbf{x}' = \begin{pmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{pmatrix} \mathbf{x}.$$

Hints - There are three linearly independent eigenvectors, and one of the eigenvalues is 8.

Solution - The characteristic polynomial for the matrix is

$$\begin{vmatrix} 3-\lambda & 2 & 4 \\ 2 & -\lambda & 2 \\ 4 & 2 & 3-\lambda \end{vmatrix} = -(\lambda - 8)(\lambda + 1)^2.$$

So, the eigenvalues are $\lambda = 8, -1, -1$.

The eigenvector for $\lambda = 8$ is:

$$\begin{pmatrix} -5 & 2 & 4 \\ 2 & -8 & 2 \\ 4 & 2 & -5 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}.$$

For $\lambda = -1$ the eigenvectors are:

$$\begin{pmatrix} 5 & 2 & 4 \\ 2 & 1 & 2 \\ 4 & 2 & 4 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \quad \text{or} \quad \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 1 \\ -4 \\ 1 \end{pmatrix}.$$

So, even though there is a repeated eigenvalue, we still have three linearly independent eigenvectors, and our general solution is

$$\mathbf{x}(t) = c_1 \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} e^{8t} + c_2 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} e^{-t} + c_3 \begin{pmatrix} 1 \\ -4 \\ 1 \end{pmatrix} e^{-t}.$$

6. **Convolutions and Laplace Transforms** (10 points)

Find the inverse Laplace transform of the function

$$F(s) = \frac{s}{(s-3)(s^2+1)}.$$

Hint - $\displaystyle\int e^{-3\tau}\cos(\tau)d\tau = \frac{1}{10}\left(e^{-3\tau}\sin(\tau) - 3e^{-3\tau}\cos(\tau)\right).$

Solution - The function $F(s)$ is the product

$$\left(\frac{1}{s-3}\right)\left(\frac{s}{s^2+1}\right) = \mathcal{L}(e^{3t})\mathcal{L}(\cos(t)).$$

The inverse Laplace transform of this product will be the convolution

$$\begin{aligned} e^{3t} * \cos(t) &= \int_0^t e^{3(t-\tau)}\cos(\tau)d\tau = e^{3t}\int_0^t e^{-3\tau}\cos(\tau)d\tau \\ &= \frac{e^{3t}}{10}\left(e^{-3\tau}\sin(\tau) - 3e^{-3\tau}\cos(\tau)\right)\Big|_0^t \\ &= \frac{1}{10}\left(\sin(t) - 3\cos(t) + 3e^{3t}\right). \end{aligned}$$

7. **Power Series Solutions** (15 points)

Use the power series method to find the first six terms (up to the x^6 term) in a power series solution to the differential equation

$$3y'' + xy' - 4y = 0.$$

Solution - Setting our solution up as a power series

$$y(x) = \sum_{n=0}^{\infty} c_n x^n$$

and plugging this into our differential equation we get

$$3\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + \sum_{n=1}^{\infty} nc_n x^n - 4\sum_{n=0}^{\infty} c_n x^n = 0.$$

If we shift the first sum by 2, and note the second sum will be the same if we begin at $n = 0$, we get

$$\sum_{n=0}^{\infty}(3(n+2)(n+1)c_{n+2} + (n-4)c_n)x^n = 0.$$

Applying the identity principle we get the recursion relation

$$c_{n+2} = \frac{4-n}{3(n+2)(n+1)}c_n.$$

For the even terms we get:

$$c_0 = c_0;$$

$$c_2 = \frac{2}{3}c_0;$$

$$c_4 = \frac{1}{18}c_2 = \frac{1}{27}c_0;$$

$$c_6 = 0.$$

All higher even terms will also be 0. As for the odd terms we get:

$$c_1 = c_1;$$

$$c_3 = \frac{1}{6}c_1;$$

$$c_5 = \frac{1}{60}c_3 = \frac{1}{360}c_1.$$

So, up to the x^6 term our solution is

$$y(x) = c_0\left(1 + \frac{2}{3}x^2 + \frac{1}{27}x^4\right) + c_1\left(x + \frac{1}{6}x^3 + \frac{1}{360}x^5 + \cdots\right).$$

In case you're curious, the general solution is

$$y(x) = c_0\left(1 + \frac{2}{3}x^2 + \frac{1}{27}x^4\right) +$$
$$c_1\left(x + \frac{1}{6}x^3 + \frac{1}{360}x^5 + 3\sum_{n=3}^{\infty}\frac{(-1)^n(2n-5)!!x^{2n+1}}{(2n+1)!3^n}\right).$$

8. **Ordinary Points, Regular Singular Points, and Irregular Singular Points** (5 points)

   Determine if the point $x = 0$ is an ordinary point, a regular singular point, or an irregular singular point for the linear differential equation

$$x^3 y'' - xy' + 4xy = 0.$$

   Solution - We can rewrite this differential equation as

$$y'' - \frac{1}{x^2} y' + \frac{4}{x^2} y = 0.$$

   The coefficient functions $P(x) = -1/x^2$ and $Q(x) = 4/x^2$ are both singular at $x = 0$, so it's a singular point. The functions

$$p(x) = xP(x) = -\frac{1}{x},$$

$$q(x) = x^2 Q(x) = 4$$

   are not both nonsingular at $x = 0$, as $p(x)$ is singular. So, $x = 0$ is an *irregular singular point*.

9. **Endpoint Value Problems** (10 points)

Find the eigenvalues and eigenfunctions corresponding to the nontrivial solutions of the endpoint value problem

$$X''(x) + \lambda X(x) = 0,$$
$$X(0) = X(2) = 0.$$

Solution - If $\lambda < 0$ then our solution will be of the form

$$X(x) = Ae^{\sqrt{-\lambda}x} + Be^{-\sqrt{-\lambda}x}.$$

Plugging in our boundary values gives us

$$X(0) = A + B = 0 \Rightarrow A = -B,$$

and

$$X(2) = Ae^{2\sqrt{-\lambda}} + Be^{-2\sqrt{-\lambda}} = A(e^{2\sqrt{-\lambda}} - e^{-2\sqrt{-\lambda}}) = 0.$$

If $A \neq 0$ then we must have

$$e^{2\sqrt{-\lambda}} = e^{-2\sqrt{-\lambda}}.$$

The only point where $e^x = e^{-x}$ is at $x = 0$, and as $\lambda < 0$ this cannot be the case. So, $\lambda < 0$ has no nontrivial solution.

If $\lambda = 0$ then the solution to the ODE is

$$X(x) = Ax + B.$$

If we plug in our boundayr values we get

$$X(0) = B = 0,$$

$$X(2) = 2A = 0 \Rightarrow A = 0.$$

So, there is no nontrivial solution, and $\lambda = 0$ is not an eigenvalue.

Finally, if $\lambda > 0$ the solution to the ODE is

$$X(x) = A\cos(\sqrt{\lambda}x) + B\sin(\sqrt{\lambda}x).$$

If we plug in our boundary values we get

$$X(0) = A = 0,$$

$$X(2) = B\sin(2\sqrt{\lambda}) = 0.$$

If $B \neq 0$ we must have $\sin(2\sqrt{\lambda}) = 0$. As $\sin(x) = 0$ if $x = n\pi$ this would imply

$$2\sqrt{\lambda} = n\pi \Rightarrow \lambda = \frac{n^2\pi^2}{4}.$$

So, the eigenvalues are

$$\lambda_n = \frac{n^2\pi^2}{4},$$

and the corresponding eigenfunctions are

$$X_n(x) = \sin\left(\frac{n\pi x}{2}\right).$$

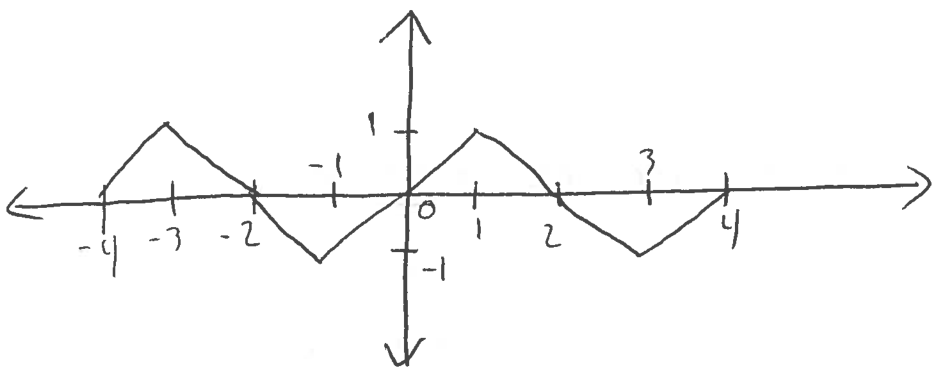
10. **Fourier Series** (15 points)

Graph the odd extension of the function

$$f(x) = \begin{cases} x & 0 < x < 1 \\ 2 - x & 1 \leq x < 2 \end{cases}$$

and find its Fourier sine series.

Solution -



The Fourier coefficients will be:

$$B_n = \frac{2}{2}\int_0^2 f(x)\sin\left(\frac{n\pi x}{2}\right)dx =$$

$$\int_0^1 x\sin\left(\frac{n\pi x}{2}\right) + \int_1^2 (2-x)\sin\left(\frac{n\pi x}{2}\right)$$

$$= \left(\frac{4}{n^2\pi^2}\sin\left(\frac{n\pi x}{2}\right) - \frac{2}{n\pi}x\cos\left(\frac{n\pi x}{2}\right)\right)\Big|_0^1 - \frac{4}{n\pi}\cos\left(\frac{n\pi x}{2}\right)\Big|_1^2 -$$

$$\left(\frac{4}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right) - \frac{2}{n\pi}x\cos\left(\frac{n\pi x}{2}\right)\right)\Big|_1^2$$

$$= \frac{4}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right) - \frac{2}{n\pi}\cos\left(\frac{n\pi}{2}\right) - \frac{4}{n\pi}\cos(n\pi) + \frac{4}{n\pi}\cos\left(\frac{n\pi}{2}\right) +$$

$$\frac{4}{n\pi}\cos(n\pi) + \frac{4}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right) - \frac{2}{n\pi}\cos\left(\frac{n\pi}{2}\right)$$

$$= \frac{8}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right) = \begin{cases} (-1)^{\frac{n-1}{2}}\left(\frac{8}{n^2\pi^2}\right) & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$$

So, the Fourier sine series is

$$\frac{8}{\pi^2}\left(\sin\left(\frac{\pi x}{2}\right) - \frac{1}{3^2}\sin\left(\frac{3\pi x}{2}\right) + \frac{1}{5^2}\sin\left(\frac{5\pi x}{2}\right) - \frac{1}{7^2}\sin\left(\frac{7\pi x}{2}\right) + \cdots\right).$$

11. **The Heat Equation** (10 points)

Find the solution to the partial differential equation

$$u_t = 3u_{xx},$$

$$u(0,t) = u(2,t) = 0,$$

$$u(x,0) = \begin{cases} x & 0 < x < 1 \\ 2 - x & 1 \leq x < 2 \end{cases}$$

Solution - We assume our solution is a separable equation

$$u(x,t) = X(x)T(t).$$

Plugging this into our differential equation we get

$$X(x)T'(t) = 3X''(x)T(t)$$

$$\Rightarrow \frac{T'(t)}{3T(t)} = \frac{X''(x)}{X(x)} = -\lambda.$$

Where $-\lambda$ is a constant by our standard argument. This means the function $X(x)$ must satisfy the differential equation

$$X''(x) + \lambda X(x) = 0,$$

with the endpoint conditions $X(0) = X(2) = 0$. As we saw derived in Problem 9, this means X will be a function of the form

$$X_n(x) = \sin\left(\frac{n\pi x}{2}\right),$$

with corresponding eigenvalues

$$\lambda_n = \frac{n^2\pi^2}{4}.$$

The function T_n must satisfy the differential equation

$$T_n'(t) + 3\lambda_n T_n(t) = 0.$$

This differential equation has the solution

$$T_n(t) = Ce^{-3\lambda_n t} = Ce^{-\frac{3n^2\pi^2}{4}t}.$$

The corresponding solutions to the PDE will be

$$u_n(x,t) = A_n \sin\left(\frac{n\pi x}{2}\right)e^{-\frac{3n^2\pi^2 t}{4}}.$$

We need to find coefficients A_n such that

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{2}\right)e^{-\frac{3n^2\pi^2}{4}}$$

satisfies

$$u(x,0) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{2}\right) = \begin{cases} x & 0 < x < 1 \\ 2-x & 1 \le x < 2 \end{cases}$$

on the interval $0 < x < 2$. To do this, we take the odd extension of $u(x,0)$, and find the corresponding Fourier coefficients for the odd extension. We already did this in Problem 10, and the solution is

$$A_n = \begin{cases} (-1)^{\frac{n-1}{2}}\left(\frac{8}{n^2\pi^2}\right) & n\ odd \\ 0 & n\ even \end{cases}$$

So, our solution is

$$u(x,t) = \frac{8}{\pi^2}\sum_{n=0}^{\infty}(-1)^n \sin\left(\frac{(2n+1)\pi}{2}\right)e^{-\frac{3(2n+1)^2\pi^2 t}{4}}$$

$$=$$

$$\frac{8}{\pi^2}\left(\sin\left(\frac{\pi x}{2}\right)e^{-\frac{3\pi^2 t}{4}} - \frac{1}{3^2}\sin\left(\frac{3\pi x}{2}\right)e^{-\frac{12\pi^2 t}{4}} + \frac{1}{5^2}\sin\left(\frac{5\pi x}{2}\right)e^{-\frac{75\pi^2 t}{4}} - \cdots\right)$$

12. **Nonlinear Systems of ODEs** (10 Points Extra Credit)

Find all the critical points of the system

$$\frac{dx}{dt} = y^2 - 1,$$

$$\frac{dy}{dt} = x^3 - y,$$

and determine if each critical point is either stable or unstable.

Solution - The Jacobian matrix for this system of differential equations is

$$J(x, y) = \begin{pmatrix} 0 & 2y \\ 3x^2 & -1 \end{pmatrix}.$$

The critical points will be:

$$y^2 - 1 = 0 \Rightarrow y = \pm 1;$$

$$x^3 - y = 0 \Rightarrow x^3 = y.$$

If $y = 1$ then $x = 1$, and if $y = -1$ then $x = -1$. So, the two critical points are $(1, 1)$ and $(-1, -1)$.

As for the stability of these critical points, the values of our Jacobian matrix at these points are

$$J(1, 1) = \begin{pmatrix} 0 & 2 \\ 3 & -1 \end{pmatrix}, \quad J(-1, -1) = \begin{pmatrix} 0 & -2 \\ 3 & -1 \end{pmatrix}.$$

The eigenvalues for these will be:

$$\begin{vmatrix} -\lambda & 2 \\ 3 & -1-\lambda \end{vmatrix} = \lambda(\lambda+1) - 6 = \lambda^2 + \lambda - 6 = (\lambda+3)(\lambda-2),$$

$$\text{so } \lambda = 2, -3.$$

$$\begin{vmatrix} -\lambda & -2 \\ 3 & -1-\lambda \end{vmatrix} = \lambda(\lambda+1) + 6 = \lambda^2 + \lambda + 6,$$

$$\text{so } \lambda = \tfrac{-1 \pm \sqrt{1-24}}{2} = \tfrac{-1 \pm i\sqrt{23}}{2}.$$

At $(1, 1)$ we have two real eigenvalues of different sign, so it's a saddle point, which is unstable. At $(-1, -1)$ we have complex eigenvalues with a negative real part, so we have a stable spiral point.