Math 2280 - Final Exam

University of Utah

Fall 2013

Name: ____

This is a 2 hour exam. Please show all your work, as a worked problem is required for full points, and partial credit may be rewarded for some work in the right direction.

Things You Might Want to Know

Definitions

$$\mathcal{L}(f(t)) = \int_0^\infty e^{-st} f(t)dt.$$

$$f(t) * g(t) = \int_0^t f(\tau)g(t-\tau)d\tau.$$

Laplace Transforms

$$\mathcal{L}(t^n) = \frac{n!}{s^{n+1}}$$

$$\mathcal{L}(e^{at}) = \frac{1}{s-a}$$

$$\mathcal{L}(\sin(kt)) = \frac{k}{s^2+k^2}$$

$$\mathcal{L}(\cos(kt)) = \frac{s}{s^2+k^2}$$

$$\mathcal{L}(\delta(t-a)) = e^{-as}$$

$$\mathcal{L}(u(t-a)f(t-a)) = e^{-as}F(s).$$

Translation Formula

$$\mathcal{L}(e^{at}f(t)) = F(s-a).$$

Derivative Formula

$$\mathcal{L}(x^{(n)}) = s^n X(s) - s^{n-1}x(0) - s^{n-2}x'(0) - \cdots - sx^{(n-2)}(0) - x^{(n-1)}(0).$$

Fourier Series Definition

For a function $f(t)$ of period $2L$ the Fourier series is:

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi t}{L}\right) + b_n \sin\left(\frac{n\pi t}{L}\right) \right).$$

$$a_n = \frac{1}{L} \int_{-L}^{L} f(t) \cos\left(\frac{n\pi t}{L}\right) dt$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(t) \sin\left(\frac{n\pi t}{L}\right) dt.$$

1. **Basic Definitions** (5 points)

   (a) (3 points) State the order of the differential equation

   $$e^x y^{(3)} - 2\sin(y) + 3x^4 y' = \ln x.$$

   (b) (2 points) Is the differential equation

   $$(x+1)y^{(2)} + 2y = 0$$

   linear or nonlinear?

2. **Undetermined Coefficients** (5 points)

   Use the method of undetermined coefficients to state the form of the particular solution to the differential equation

   $$y^{(3)} - y'' - 4y' + 4y = x^2 e^{2x} \sin(3x).$$

   You do not have to solve for the coefficients, or solve the differential equation.

3. **Converting to a First-Order System** (5 points)

   Convert the differential equation

$$y^{(3)} - y'' - 4y' + 4y = x^2 e^{2x} \sin(3x).$$

   into an equivalent system of first-order equations.

4. **Linear ODEs with Constant Coefficients** (5 points)

   Find the general solution to the homogeneous equation corresponding to the differential equation

$$y^{(3)} - y'' - 4y' + 4y = x^2 e^{2x} \sin(3x).$$

   Hint - One root of the polynomial $r^3 - r^2 - 4r + 4$ is $r = 2$.

5. **Solving Systems of Linear ODEs** (15 points)

   Find the general solution to the system of ODEs

$$\mathbf{x}' = \begin{pmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{pmatrix} \mathbf{x}.$$

   Hints - There are three linearly independent eigenvectors, and one of the eigenvalues is 8.

More room for Problem 5, if you need it.

6. **Convolutions and Laplace Transforms** (10 points)
Find the inverse Laplace transform of the function

$$F(s) = \frac{s}{(s-3)(s^2+1)}.$$

Hint - $\displaystyle\int e^{-3\tau}\cos(\tau)d\tau = \frac{1}{10}\left(e^{-3\tau}\sin(\tau) - 3e^{-3\tau}\cos(\tau)\right).$

More room for Problem 6, if you need it.

7. **Power Series Solutions** (15 points)

   Use the power series method to find the first six terms (up to the x^6 term) in a power series solution to the differential equation

$$3y'' + xy' - 4y = 0.$$

More room for Problem 7, if you need it.

8. **Ordinary Points, Regular Singular Points, and Irregular Singular Points** (5 points)

   Determine if the point $x = 0$ is an ordinary point, a regular singular point, or an irregular singular point for the linear differential equation

$$x^3 y'' - xy' + 4xy = 0.$$

9. **Endpoint Value Problems** (10 points)

   Find the eigenvalues and eigenfunctions corresponding to the non-trivial solutions of the endpoint value problem

$$X''(x) + \lambda X(x) = 0,$$

$$X(0) = X(2) = 0.$$

More room for Problem 9, if you need it.

10. **Fourier Series** (15 points)

Graph the odd extension of the function

$$f(x) = \begin{cases} x & 0 < x < 1 \\ 2 - x & 1 \leq x < 2 \end{cases}$$

and find its Fourier sine series.

More room for Problem 10, if you need it.

11. **The Heat Equation** (10 points)

Find the solution to the partial differential equation

$$u_t = 3u_{xx},$$

$$u(0,t) = u(2,t) = 0,$$

$$u(x,0) = \begin{cases} x & 0 < x < 1 \\ 2 - x & 1 \leq x < 2 \end{cases}$$

More room for Problem 11, if you need it.

12. **Nonlinear Systems of ODEs** (10 Points Extra Credit)

Find all the critical points of the system

$$\frac{dx}{dt} = y^2 - 1,$$

$$\frac{dy}{dt} = x^3 - y,$$

and determine if each critical point is either stable or unstable.

More room for Problem 12, if you need it.