Math 2280 - Exam 3

University of Utah

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This is a 50 minute exam. Please show all your work, as a worked problem is required for full points, and partial credit may be rewarded for some work in the right direction.

Things You Might Want to Know

Definitions

$$\mathcal{L}(f(t)) = \int_0^\infty e^{-st} f(t)dt.$$

$$f(t) * g(t) = \int_0^t f(\tau)g(t-\tau)d\tau.$$

Laplace Transforms

$$\mathcal{L}(t^n) = \frac{n!}{s^{n+1}}$$

$$\mathcal{L}(e^{at}) = \frac{1}{s-a}$$

$$\mathcal{L}(\sin(kt)) = \frac{k}{s^2+k^2}$$

$$\mathcal{L}(\cos(kt)) = \frac{s}{s^2+k^2}$$

$$\mathcal{L}(\delta(t-a)) = e^{-as}$$

$$\mathcal{L}(u(t-a)f(t-a)) = e^{-as}F(s).$$

Translation Formula

$$\mathcal{L}(e^{at}f(t)) = F(s-a).$$

Derivative Formula

$$\mathcal{L}(x^{(n)}) = s^n X(s) - s^{n-1}x(0) - s^{n-2}x'(0) - \cdots - sx^{(n-2)}(0) - x^{(n-1)}(0).$$

1. (15 Points) *Calculating a Laplace Transform*

   Calculate the Laplace transform of the function

   $$f(t) = t - 4$$

   using the formal definition.

   Solution - The Laplace transform is:

   $$\mathcal{L}(f(t)) = \int_0^\infty (t-4)e^{-st}dt = \int_0^\infty te^{-st}dt - 4\int_0^\infty e^{-st}dt$$

   $$= \left(-\frac{te^{-st}}{s} - \frac{e^{-st}}{s^2}\right)\Big|_0^\infty - 4\left(-\frac{e^{-st}}{s}\right)\Big|_0^\infty = \frac{1}{s^2} - \frac{4}{s}.$$

   $$s > 0.$$

2. (15 Points) *Convolutions*

Calculate the the convolution $f(t) * g(t)$ of the following functions:

$$f(t) = t \quad g(t) = \cos(t).$$

Solution - The convolution is:

$$f(t) * g(t) = \int_0^t (t-\tau)\cos(\tau)d\tau = \left(t\sin(\tau) - \tau\sin(\tau) - \cos(\tau)\right)\Big|_0^t$$

$$= t\sin(t) - t\sin(t) - \cos(t) + 1 = 1 - \cos(t).$$

3. (30 Points) *Delta Functions and Laplace Transforms*

Solve the initial value problem

$$x'' + 4x' + 4x = 1 + \delta(t-2).$$
$$x(0) = x'(0) = 0.$$

The Laplace transforms of x, x', and x'' are:

$$\mathcal{L}(x) = X(s);$$
$$\mathcal{L}(x') = sX(s) - x(0) = sX(s);$$
$$\mathcal{L}(x'') = s^2X(s) - sx(0) - x'(0) = s^2X(s).$$

So, the Laplace transform of the left-hand side of the differential equation is:

$$s^2X(s) + 4sX(s) + 4X(s) = (s+2)^2X(s).$$

The Laplace transform of the right-hand side of the differential equation is:

$$\frac{1}{s} + e^{2s}.$$

So, taking the Laplace transform of both sides of the differential equation and equating them we get:

$$(s+2)^2X(s) = \frac{1}{s} + e^{2s}$$
$$\Rightarrow X(s) = \frac{1}{s(s+2)^2} + \frac{e^{2s}}{(s+2)^2}.$$

A partial-fraction decomposition of the first term on the right gives:

$$\frac{1}{s(s+2)^2} = \frac{A}{s} + \frac{B}{s+2} + \frac{C}{(s+2)^2} = \frac{A(s+2)^2 + Bs(s+2) + Cs}{s(s+2)^2}$$
$$= \frac{(A+B)s^2 + (4A+2B+C)s + 4A}{s(s+2)^2}.$$

From this we get

$$A = \frac{1}{4},$$
$$B = -A = -\frac{1}{4},$$
$$C = -4A - 2B = -\frac{1}{2}.$$

So, the partial fraction decomposition is:

$$X(s) = \frac{1}{s(s+2)^2} + \frac{e^{2s}}{(s+2)^2}$$
$$= \frac{1}{4}\left(\frac{1}{s}\right) - \frac{1}{4}\left(\frac{1}{s+2}\right) - \frac{1}{2}\left(\frac{1}{(s+2)^2}\right) + \frac{e^{2s}}{(s+2)^2}.$$

From this we get that the inverse Laplace transform is:

$$x(t) = \frac{1}{4} - \frac{1}{4}e^{-2t} - \frac{1}{2}te^{-2t} + u(t-2)(t-2)e^{-2(t-2)}.$$

4. (10 Points) *Singular Points*

Determine whether the point $x = 0$ is an ordinary point, a regular singular point, or an irregular singular point of the differential equation:

$$x^2(1-x^2)y'' + 2xy' - 2y = 0.$$

Solution - We can rewrite the differential equation as:

$$y'' + \frac{2}{x(1-x^2)} - \frac{2}{x^2(1-x^2)} = 0.$$

From this we see $x = 0$ is a singular point. If we analyze the functions:

$$p(x) = x\left(\frac{2}{x(1-x^2)}\right) = \frac{2}{1-x^2};$$
$$q(x) = x^2\left(\frac{2}{x^2(1-x^2)}\right) = \frac{2}{1-x^2}.$$

We see for both that $x = 0$ is an ordinary point, and therefore the point $x = 0$ is a regular singular point of the above differential equation.

5. (30 points) *Power Series*

Use power series methods to find the general solution to the differential equation:

$$(x^2 + 2)y'' + 4xy' + 2y = 0.$$

State the recurrence relation and the guaranteed radius of convergence.

We can rewrite the differential equation as:

$$y'' + \frac{4x}{x^2+2}y' + \frac{2}{x^2+2}y = 0.$$

The point $x = 0$ is an ordinary point, and the singular points are at $x = \pm\sqrt{2}i$. Both points have a distance $\sqrt{2}$ from the origin, so the guaranteed radius of convergence is $\sqrt{2}$.

A power series solution $y(x)$ and its derivatives will have the following forms:

$$y(x) = \sum_{n=0}^{\infty} c_n x^n;$$

$$y'(x) = \sum_{n=1}^{\infty} n c_n x^{n-1};$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2}.$$

If we plug these into our differential equation we get:

$$\sum_{n=2}^{\infty} n(n-1)c_n x^n + 2\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + 4\sum_{n=1}^{\infty} nc_n x^n + 2\sum_{n=0}^{\infty} c_n x^n = 0.$$

We can rewrite this as:

$$\sum_{n=0}^{\infty}(2(n+2)(n+1)c_{n+2} + (n^2+3n+2)c_n)x^n = 0.$$

From this the identity principle gives us the recurrence relation:

$$c_{n+2} = -\frac{n^2+3n+2}{2(n+2)(n+1)}c_n = -\frac{(n+2)(n+1)}{2(n+2)(n+1)}c_n = -\frac{1}{2}c_n.$$

This is the recurrence relation for a geometric series, where c_0 and c_1 are arbitrary, and our solution is:

$$y(x) = c_0\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{2^n} + c_1 x\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{2^n} = \frac{2c_0}{2-x^2} + \frac{2c_1 x}{2-x^2}.$$