

Math 2280 - Exam 2

University of Utah

Fall 2013

Name: *Solutions by Dylan Zwick*

This is a 50 minute exam. Please show all your work, as a worked problem is required for full points, and partial credit may be rewarded for some work in the right direction.

1. (10 Points) *Taking the Lower (Order) Road*

Convert the differential equation

$$x^{(4)} - 4x^{(3)} + 2x'' + 5x' - 7x = \cos 2t$$

into an equivalent system of first-order equations.

Solution - We define variables x_1, x_2, x_3, x_4 as:

$$x = x_1,$$

$$x_1' = x_2,$$

$$x_2' = x_3,$$

$$x_3' = x_4.$$

So, our first-order system of equations is:

$$x_1' = x_2$$

$$x_2' = x_3$$

$$x_3' = x_4$$

$$x_4' = 4x_4 - 2x_3 - 5x_2 + 7x_1 + \cos 2t.$$

2. (15 Points) *Take a Wild Guess*

Use the method of undetermined coefficients to find the form of the particular solution to the differential equation:

$$y'' - 4y' + 3y = x^2 \cos(2x)e^{2x}$$

Solution - The corresponding homogeneous equation is:

$$y'' - 4y' + 3y = 0,$$

which has characteristic equation:

$$r^2 - 4r + 3.$$

The roots of this characteristic equation are $r = 1, 3$, and so the homogeneous solution will have the form:

$$y(x) = c_1 e^x + c_2 e^{3x}.$$

The function $f(x) = x^2 \cos(2x)e^{2x}$ is the product of a second-order polynomial, a trigonometric function, and an exponential, and so the “guess” for the particular solution will be the same, where we note we need to include both sines and cosines in the guess:

$$y_p = (Ax^2 + Bx + C) \cos(2x)e^{2x} + (Dx^2 + Ex + F) \sin(2x)e^{2x}.$$

All the terms in this guess are linearly independent of the homogeneous solution, so it's good to go.

3. (25 Points) *Linear Differential Equations*

Find the solution to the initial value problem

$$x^{(3)} - 2x'' + x' = 0$$

with initial conditions $x(0) = 3, x'(0) = 2, x''(0) = 3$.

Solution - The characteristic polynomial for the differential equation is

$$r^3 - 2r^2 + r = 0.$$

This polynomial factors as $r(r-1)^2$, and so has roots $r = 0, 1, 1$, where I write 1 twice to illustrate that it is a root of multiplicity 2.

The corresponding homogeneous solution will have the form:

$$x(t) = c_1 + c_2e^t + c_3te^t.$$

The first and second derivatives of this solution are:

$$x'(t) = (c_2 + c_3)e^t + c_3te^t,$$

$$x''(t) = (c_2 + 2c_3)e^t + c_3te^t.$$

Plugging in our initial conditions we have the system of equations:

$$3 = c_1 + c_2,$$

$$2 = c_2 + c_3,$$

$$3 = c_2 + 2c_3.$$

Solving this system for c_1, c_2 , and c_3 we get $c_1 = 2, c_2 = 1, c_3 = 1$. So, the solution to our initial value problem is:

$$x(t) = 2 + e^t + te^t.$$

4. (25 Points) *Solving First-Order Systems*

Find the general solution to the series of first-order differential equations¹

$$x'_1 = x_1 + 2x_2 + 2x_3,$$

$$x'_2 = 2x_1 + 7x_2 + x_3,$$

$$x'_3 = 2x_1 + x_2 + 7x_3.$$

Solution - The eigenvalues of the coefficient matrix are:

$$\begin{vmatrix} 1 - \lambda & 2 & 2 \\ 2 & 7 - \lambda & 1 \\ 2 & 1 & 7 - \lambda \end{vmatrix} \\ = (1 - \lambda)(7 - \lambda)^2 + 4 + 4 - (1 - \lambda) - 4(7 - \lambda) - 4(7 - \lambda) \\ = -\lambda(\lambda - 9)(\lambda - 6).$$

So, the eigenvalues for this matrix are $\lambda = 0, 6, 9$.

For the eigenvalue $\lambda = 0$ an eigenvector must satisfy:

$$\begin{pmatrix} 1 & 2 & 2 \\ 2 & 7 & 1 \\ 2 & 1 & 7 \end{pmatrix} \mathbf{v} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

The vector $\begin{pmatrix} 4 \\ -1 \\ -1 \end{pmatrix}$ works.

For the eigenvalue $\lambda = 6$ an eigenvector must satisfy:

¹Hint - All the eigenvalues are integers. And one of them is zero. If this is not what you're getting, check your algebra.

$$\begin{pmatrix} -5 & 2 & 2 \\ 2 & 1 & 1 \\ 2 & 1 & 1 \end{pmatrix} \mathbf{v} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

The vector $\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$ works.

For the eigenvalue $\lambda = 9$ an eigenvector must satisfy:

$$\begin{pmatrix} -8 & 2 & 2 \\ 2 & -2 & 1 \\ 2 & 1 & -2 \end{pmatrix} \mathbf{v} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

The vector $\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$ works.

So, the general solution to this system of ODEs is:

$$\mathbf{x}(t) = c_1 \begin{pmatrix} 4 \\ -1 \\ -1 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} e^{6t} + c_3 \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} e^{9t}.$$

5. (25 points) *Multiple Eigenvalues*²

Find the general solution to the system of differential equations given by the matrix equation:

$$\mathbf{x}' = \begin{pmatrix} 7 & 1 \\ -4 & 3 \end{pmatrix} \mathbf{x}.$$

Solution - The eigenvalues for the coefficient matrix are:

$$\begin{vmatrix} 7 - \lambda & 1 \\ -4 & 3 - \lambda \end{vmatrix} = (7 - \lambda)(3 - \lambda) + 4 = \lambda^2 - 10\lambda + 25 = (\lambda - 5)^2.$$

So, $\lambda = 5$ is the only eigenvalue, and it's repeated. An eigenvector for this eigenvalue must satisfy:

$$\begin{pmatrix} 2 & 1 \\ -4 & -2 \end{pmatrix} \mathbf{v} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The vector $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$ works, and there is no other linearly independent eigenvector. So, we want to find a chain of generalized eigenvectors. To do this we note:

$$\begin{pmatrix} 2 & 1 \\ -4 & -2 \end{pmatrix}^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

So, *any* vector that is not an eigenvector will work for the top vector in our chain. If we choose $\mathbf{v}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ we get:

²The title of this problem is a hint.

$$\mathbf{v}_1 = \begin{pmatrix} 2 & 1 \\ -4 & -2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ -4 \end{pmatrix}.$$

So, our general solution will be:

$$\mathbf{x}(t) = c_1 \begin{pmatrix} 2 \\ -4 \end{pmatrix} e^{5t} + c_2 \left(\begin{pmatrix} 2 \\ -4 \end{pmatrix} t + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right) e^{5t}.$$