Math 2280 - Exam 1

University of Utah

Fall 2013

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This is a 50 minute exam. Please show all your work, as a worked problem is required for full points, and partial credit may be rewarded for some work in the right direction.

1. (20 Points) *Differential Equation Basics*

(a) (5 points) What is the order of the differential equation given below?[1]

$$x^5y^{(4)} + (e^{x^2} + 7x^3)y^{(3)} - \sin(y^{(5)}) + x^2y' = y + x^2 - 2x + 7y^{(2)}$$

Solution : **5**.

(b) (5 points) Is the differential equation given below linear?

$$x^2y^{(3)} - 2xy' + e^x = \sin(x)y''$$

Solution : **Yes**.

(c) (10 points) On what intervals are we guaranteed a unique solution exists for the differential equation below?

$$y' + e^x y = \frac{x+2}{x-1}$$

Solution - The function e^x is continuous everywhere, while the function $\frac{x+2}{x-1}$ is continuous at all points except $x = 1$. So, there will exist a unique solution on the entire interval for any initial condition with x-value in the interval $(-\infty, 1)$ or $(1, \infty)$.

[1] Extra credit - Solve this differential equation! Just kidding. Do not attempt to solve it.

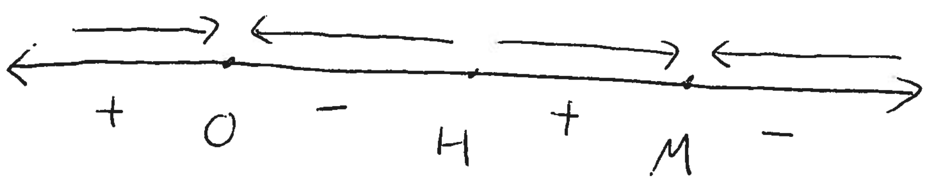
2. (10 points) *Phase Diagrams*

Find the critical points for the autonomous equation:

$$\frac{dP}{dt} = kP(M-P)(P-H),$$

where $k, M, H > 0$ and $M > H$. Draw the corresponding phase diagram, and indicate if the critical points are stable, unstable, or semistable.

Solution - The function $kP(M-P)(P-H)$ has roots (zeros) at the points $P = 0, H, M$. So, those are the critical points. The corresponding phase diagram is:



From this we see that 0 and M are stable critical points, while H is unstable.

3. (20 Points) *Separable Equations*

   Find the solution to the initial value problem:

$$\frac{dy}{dx} = 2xe^{x^2-y}$$
$$y(0) = 0.$$

Solution - We can rewrite the equation as:

$$\frac{dy}{dx} = 2xe^{x^2}e^{-y}.$$

In this form we can see it's a separable differential equation. Getting all the y terms on the left side and the x terms on the right gives us:

$$e^y dy = 2xe^{x^2}dx.$$

Integrating both sides gives us:

$$e^y = e^{x^2} + C.$$

Taking the natural logarithm of both sides we get:

$$y(x) = \ln(e^{x^2} + C).$$

Plugging in $x = 0$ we get $y(0) = \ln(1 + C) = 0$, from which we get $C = 0$. So, our solution is:

$$y(x) = \ln(e^{x^2}) = x^2.$$

4. (15 points) *Exact Equations*

Find the solution to the initial value problem[2]:

$$\frac{dy}{dx} = -\frac{\cos(x) + ye^x}{e^x + 2y},$$

$$y(0) = 2.$$

Solution - Taking our cue from the title of this problem, we can rewrite this ODE as:

$$(\cos(x) + ye^x)dx + (e^x + 2y)dy = 0.$$

We check that this equation is exact:

$$\frac{\partial}{\partial y}(\cos(x) + ye^x) = e^x = \frac{\partial}{\partial x}(e^x + 2y).$$

So, the equation is exact. We want to find an equation $F(x, y)$ such that:

$$\frac{\partial F}{\partial x} = \cos(x) + ye^x,$$

and

$$\frac{\partial F}{\partial y} = e^x + 2y.$$

For this to be true we must have:

[2]The title of this problem is a hint.

$$F(x,y) = \int (\cos(x) + ye^x)dx = \sin(x) + ye^x + g(y).$$

Solving this for $g(y)$ we get:

$$\frac{\partial F}{\partial y} = e^x + g'(y) = e^x + 2y.$$

So, $g'(y) = 2y$, and therefore $g(y) = y^2$. So, our solution is:

$$\sin(x) + ye^x + y^2 = C.$$

Plugging in $y(0) = 2$ we get:

$$\sin(0) + 2e^0 + 2^2 = 6 = C.$$

So, the solution to our initial value problem is:

$$\sin(x) + ye^x + y^2 = 6.$$

5. (20 points) *First-Order Linear Equations*

   Find a solution to the initial value problem given below, and give the interval upon which you know the solution is unique.

$$y' + 2xy = 3e^{-x^2}$$
$$y(0) = 4.$$

   Solution - This is a linear first-order differential equation with integrating factor:

$$\rho(x) = e^{\int 2x dx} = e^{x^2}.$$

   Multiplying both sides of the ODE by this integrating factor gives us:

$$e^{x^2}y' + 2xe^{x^2}y = 3$$
$$\Rightarrow (e^{x^2}y)' = 3.$$

   Integrating both sides of the above equation gives us:

$$e^{x^2}y = 3x + C \rightarrow y(x) = 3xe^{-x^2} + Ce^{-x^2}.$$

   If we plug in the initial condition $y(0) = 4$ we get:

$$y(0) = 3(0)e^0 + Ce^0 = C = 4.$$

   So, the solution to our initial value problem is:

$$y(x) = 3xe^{-x^2} + 4e^{-x^2} = (3x+4)e^{-x^2}.$$

   As both $2x$ and $3e^{-x^2}$ are continuous on the entire real line this is the unique solution on the entire real line $\mathbb{R}$.

6. (15 points) *Euler's Method*

Use Euler's method with step size $h = 1$ to estimate the solution to the initial value problem

$$\frac{dy}{dx} = x^2 + 2y - 1$$

$$y(0) = 3$$

at $x = 2$.

Solution - Using Euler's method, the estimate after the first step will be:

$$y(1) \approx 3 + 1(0^2 + 2(3) - 1) = 8.$$

The estimate after the second step will be:

$$y(2) \approx 8 + 1(1^2 + 2(8) - 1) = 24.$$

So, our estimate is $y(2) \approx 24$.