Math 2280 - Assignment 6

Dylan Zwick

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Section 3.7 - 1, 5, 10, 17, 19

Section 3.8 - 1, 3, 5, 8, 13

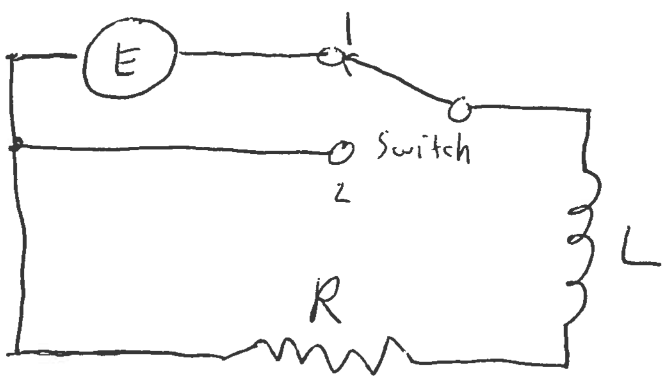
Section 4.1 - 1, 2, 13, 15, 22

Section 4.2 - 1, 10, 19, 28

Section 3.7 - Electrical Circuits

3.7.1 This problem deals with the RL circuit pictured below. It is a series circuit containing an inductor with an inductance of L henries, a resistor with a resistance of R ohms, and a source of electromotive force (emf), but no capacitor. In this case the equation governing our system is the first-order equation

$$LI' + RI = E(t).$$



Suppose that $L = 5H$, $R = 25\Omega$, and the source E of emf is a battery supplying $100V$ to the circuit. Suppose also that the switch has been in position 1 for a long time, so that a steady current of $4A$ is flowing in the circuit. At time $t = 0$, the switch is thrown to position 2, so that $I(0) = 4$ and $E = 0$ for $t \geq 0$. Find $I(t)$.

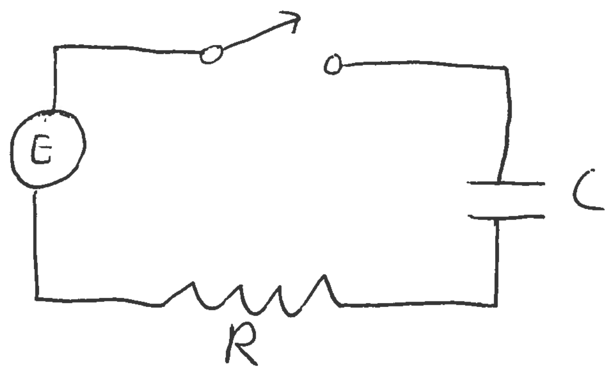
More room, if necessary, for Problem 3.7.1.

3.7.5 - In the circuit from Problem 3.7.1, with the switch in position 1, suppose that $E(t) = 100e^{-10t}\cos 60t$, $R = 20$, $L = 2$, and $I(0) = 0$. Find $I(t)$.

3.7.10 - This problem deals with an RC circuit pictured below, containing a resistor (R ohms), a capacitor (C farads), a switch, a source of emf, but no inductor. This system is governed by the linear first-order differential equation

$$R\frac{dQ}{dt} + \frac{1}{C}Q = E(t).$$

for the charge $Q = Q(t)$ on the capacitor at time t. Note that $I(t) = Q'(t)$.



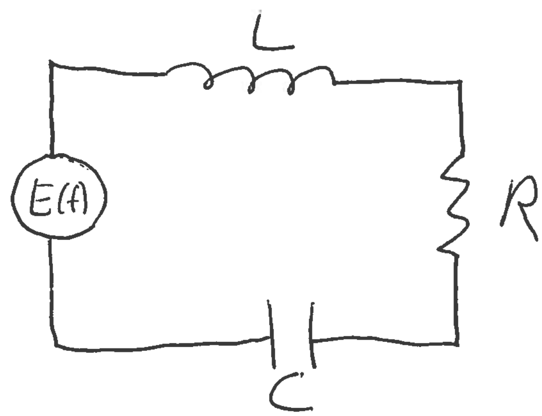
Suppose an emf of voltage $E(t) = E_0 \cos \omega t$ is applied to the RC circuit at time $t = 0$ (with the switch closed), and $Q(0) = 0$. Substitute $Q_{sp}(t) = A \cos \omega t + B \sin \omega t$ in the differential equation to show that the steady periodic charge on the capacitor is

$$Q_{sp}(t) = \frac{E_0 C}{\sqrt{1 + \omega^2 R^2 C^2}} \cos(\omega t - \beta)$$

where $\beta = \tan^{-1}(\omega RC)$.

More room for Problem 3.7.10. You'll probably need it.

3.7.17 For the RLC circuit pictured below find the current $I(t)$ using the given values of R, L, C and $V(t)$, and the given initial values.



$$R = 16\Omega, L = 2H, C = .02F;$$

$$E(t) = 100V; I(0) = 0, Q(0) = 5.$$

More room for Problem 3.7.17 if you need it.

3.7.19 Same instructions as Problem 3.7.17, but with the values:

$$R = 60\Omega, L = 2H, C = .0025F;$$

$$E(t) = 100e^{-10t}V; I(0) = 0, Q(0) = 1.$$

Section 3.8 - Endpoint Problems and Eigenvalues

3.8.1 For the eigenvalue problem

$$y'' + \lambda y = 0; \quad y'(0) = 0, y(1) = 0,$$

first determine whether $\lambda = 0$ is an eigenvalue; then find the positive eigenvalues and associated eigenfunctions.

3.8.3 Same instructions as Problem 3.8.1, but for the eigenvalue problem:

$$y'' + \lambda y = 0; \quad y(-\pi) = 0, y(\pi) = 0.$$

More room for Problem 3.8.3 if you need it.

3.8.5 Same instructions as Problem 3.8.1, but for the eigenvalue problem:

$$y'' + \lambda y = 0; \quad y(-2) = 0, y'(2) = 0.$$

More room for Problem 3.8.5 if you need it.

3.8.8 - Consider the eigenvalue problem

$$y'' + \lambda y = 0; \quad y(0) = 0 \quad y(1) = y'(1) \text{ (not a typo).;}$$

all its eigenvalues are nonnegative.

(a) Show that $\lambda = 0$ is an eigenvalue with associated eigenfunction $y_0(x) = x$.

(b) Show that the remaining eigenfunctions are given by $y_n(x) = \sin \beta_n x$, where β_n is the nth positive root of the equation $\tan z = z$. Draw a sketch showing these roots. Deduce from this sketch that $\beta_n \approx (2n+1)\pi/2$ when n is large.

More room, if necessary, for Problem 3.8.8.

3.8.13 - Consider the eigenvalue problem

$$y'' + 2y' + \lambda y = 0; \quad y(0) = y(1) = 0.$$

(a) Show that $\lambda = 1$ is not an eigenvalue.

(b) Show that there is no eigenvalue λ such that $\lambda < 1$.

(c) Show that the nth positive eigenvalue is $\lambda_n = n^2\pi^2 + 1$, with associated eigenfunction $y_n(x) = e^{-x} \sin(n\pi x)$.

More room, if necessary, for Problem 3.8.13.

Section 4.1 - First-Order Systems and Applications

4.1.1 - Transform the given differential equation into an equivalent system of first-order differential equations.

$$x'' + 3x' + 7x = t^2.$$

4.1.2 - Transform the given differential equation into an equivalent system of first-order differential equations.

$$x^{(4)} + 6x'' - 3x' + x = \cos 3t.$$

4.1.13 - Find the particular solution to the system of differential equations below. Use a computer system or graphing calculator to construct a direction field and typical solution curves for the given system.

$$x' = -2y, \quad y' = 2x; \quad x(0) = 1, y(0) = 0.$$

More room, if necessary, for Problem 4.1.13.

4.1.15 - Find the general solution to the system of differential equations below. Use a computer system or graphing calculator to construct a direction field and typical solution curves for the given system.

$$x' = \frac{1}{2}y, \quad y' = -8x.$$

More room, if necessary, for Problem 4.1.15.

4.1.22 **(a)** - Beginning with the general solution of the system from Problem 13, calculate $x^2 + y^2$ to show that the trajectories are circles.

(b) - Show similarly that the trajectories of the system from Problem 15 are ellipses of the form $16x^2 + y^2 = C^2$.

More room, if necessary, for Problem 4.1.22.

Section 4.2 - The Method of Elimination

4.2.1 - Find a general solution to the linear system below. Use a computer system or graphing calculator to construct a direction field and typical solution curves for the system.

$$\begin{aligned} x' &= -x + 3y \\ y' &= 2y \end{aligned}$$

More room for Problem 4.2.1, if you need it.

4.2.10 Find a particular solution to the given system of differential equations that satisfies the given initial conditions.

$$x' + 2y' = 4x + 5y,$$

$$2x' - y' = 3x;$$

$$x(0) = 1, y(0) = -1.$$

More room for Problem 4.2.10, if you need it.

4.2.19 Find a general solution to the given system of differential equations.

$$x' = 4x - 2y,$$

$$y' = -4x + 4y - 2z,$$

$$z' = -4y + 4z.$$

More room for Problem 4.2.19, if you need it.

4.2.28 For the system below first calculate the operational determinant to determine how many arbitrary constants should appear in a general solution. Then attempt to solve the system explicitly so as to find such a general solution.

$$\begin{array}{rcccc} (D^2+D)x & + & D^2y & = & 2e^{-t} \\ (D^2-1)x & + & (D^2-D)y & = & 0 \end{array}$$

More room for Problem 4.2.28, if you need it.