Math 2280 - Assignment 15

Dylan Zwick

Fall 2013

Section 6.1 - 1, 5, 10, 18, 30

Section 6.2 - 1, 5, 8, 15, 31

Section 6.3 - 3, 4, 5, 6, 7

Section 6.1 - Stability and the Phase Plane

In problems 6.1.1 and 6.1.5 find the critical point or points of the given autonomous system, and then match the system with its phase portrait among those given in the figures. (Figures on next page.)

6.1.1 - $\frac{dx}{dt} = 2x - y, \quad \frac{dy}{dt} = x - 3y.$

6.1.5 - $\frac{dx}{dt} = 1 - y^2 \quad \frac{dy}{dt} = x + 2y.$

6.1 Problems

In Problems 1 through 8, find the critical point or points of the given autonomous system, and thereby match each system with its phase portrait among Figs. 6.1.12 through 6.1.19.

1. $\dfrac{dx}{dt} = 2x - y, \quad \dfrac{dy}{dt} = x - 3y$

2. $\dfrac{dx}{dt} = x - y, \quad \dfrac{dy}{dt} = x + 3y - 4$

3. $\dfrac{dx}{dt} = x - 2y + 3, \quad \dfrac{dy}{dt} = x - y + 2$

4. $\dfrac{dx}{dt} = 2x - 2y - 4, \quad \dfrac{dy}{dt} = x + 4y + 3$

5. $\dfrac{dx}{dt} = 1 - y^2, \quad \dfrac{dy}{dt} = x + 2y$

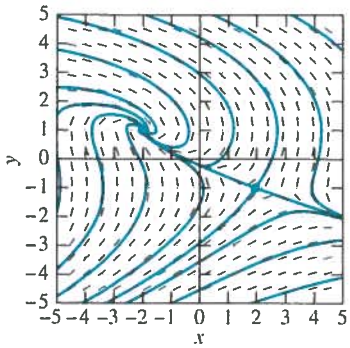

FIGURE 6.1.12. Spiral point $(-2, 1)$ and saddle point $(2, -1)$.

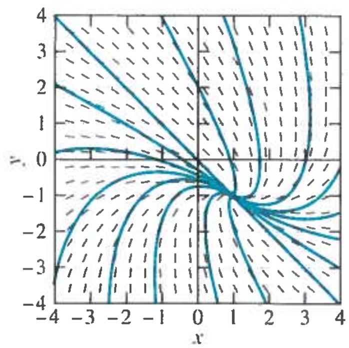

FIGURE 6.1.13. Spiral point $(1, -1)$.

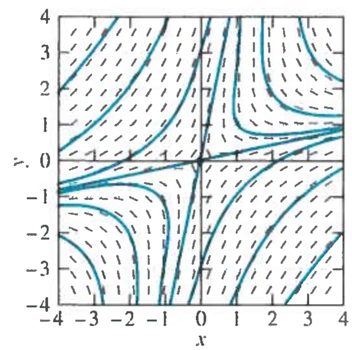

FIGURE 6.1.14. Saddle point $(0, 0)$.

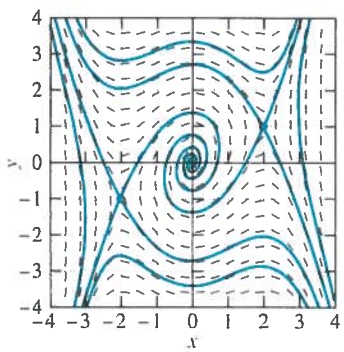

FIGURE 6.1.15. Spiral point $(0, 0)$; saddle points $(-2, -1)$ and $(2, 1)$.

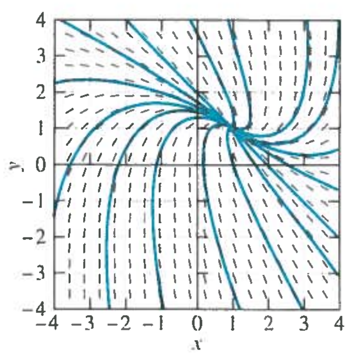

FIGURE 6.1.16. Node $(1, 1)$.

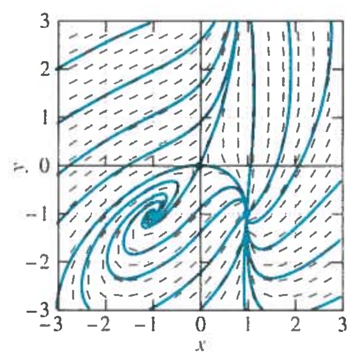

FIGURE 6.1.17. Spiral point $(-1, -1)$, saddle point $(0, 0)$, and node $(1, -1)$.

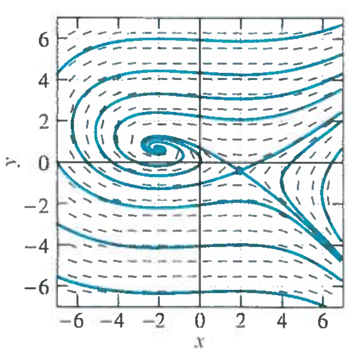

FIGURE 6.1.18. Spiral point $(-2, \frac{2}{3})$ and saddle point $(2, -\frac{2}{5})$.

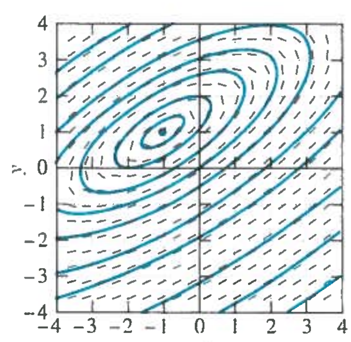

FIGURE 6.1.19. Stable center $(-1, 1)$.

6.1.10 - Find an equilibrium solution for the second-order differential equation

$$x'' + 2x' + x + 4x^3 = 0,$$

and construct a phase portrait for the equivalent first-order system

$$x' = y, y' = -f(x, y).$$

Determine whether the critical point $(x_0, 0)$ looks like a center, a saddle point, or a spiral point.

6.1.18 - Solve the system of differential equations

$$\frac{dx}{dt} = -y,$$

$$\frac{dy}{dt} = 4x,$$

and determined whether the critical point $(0, 0)$ is stable, asymptotically stable, or unstable. Construct a phase portrait for the system.

6.1.30 - Suppose the solution $(x_1(t), y_1(t))$ to the system of differential equations

$$\frac{dx}{dt} = F(x, y), \quad \frac{dy}{dt} = G(x, y),$$

is defined for all t and that its trafectory has an apparent self-intersection:

$$x_1(a) = x_1(a + P) = x_0,$$

$$y_1(a) = y_1(a + P) = y_0$$

for some $P > 0$. Introduce the solution

$$x_2(t) = x_1(t + P), \quad y_2(t) = y_1(t + P)$$

and then apply the uniqueness theorem to show that

$$x_1(t + P) = x_1(t), \quad y_1(t) = y_1(t + P)$$

for *all* t. Thus the solution $(x_1(t), y_1(t))$ is periodic with period P and has a closed trajectory. Consequently a solution of an autonomous system is either periodic with a closed trajectory, or else its trajectory never passes through the same point twice.

More room for Problem 6.1.30.

Section 6.2 - Linear and Almost Linear Systems

6.2.1 - Apply Theorem 1 from the textbook to determine the type of the critical point $(0,0)$ for the system of differential equations

$$\frac{dx}{dt} = -2x + y,$$

$$\frac{dy}{dt} = x - 2y.$$

Verify your conclusion by constructing a phase portrait for the system.

6.2.5 - Apply Theorem 1 from the textbook to determine the type of the critical point $(0,0)$ for the system of differential equations

$$\frac{dx}{dt} = x - 2y,$$

$$\frac{dy}{dt} = 2x - 3y.$$

Verify your conclusion by constructing a phase portrait for the system.

6.2.8 - Apply Theorem 1 from the textbook to determine the type of the critical point $(0, 0)$ for the system of differential equations

$$\frac{dx}{dt} = x - 3y,$$

$$\frac{dy}{dt} = 6x - 5y.$$

Verify your conclusion by constructing a phase portrait for the system.

6.2.15 - Find the critical point of the sytem of differential equations below. Then apply Theorem 2 from the textbook to classify this critical point as to type and stability. Verify your conclusion by constructing a phase portrait for the system.

$$\frac{dx}{dt} = x - y,$$

$$\frac{dy}{dt} = 5x - 3y - 2.$$

6.2.31 - For the system of differential equations

$$\frac{dx}{dt} = y^2 - 1,$$

$$\frac{dy}{dt} = x^3 - y$$

find all the critical points and investigate the type and stability of each. Verify your conclusion by constructing a phase portrait for the system.

Section 6.3 - Ecological Models: Predators and Competition

6.3.3 - Let $X(t)$ be a harmful insect population (aphids?) that under natural conditions is held somewhat in check by a benign predator insect population $y(t)$ (ladybugs?). Assume that $x(t)$ and $y(t)$ satisfy the predator-prey equations

$$\frac{dx}{dt} = ax - pxy,$$

$$\frac{dy}{dt} = -by + qxy.$$

So, the stable equilibrium populations are $x_E = b/q, y_E = a/p$. Now, suppose that an insecticide is employed that kills (per unit time) the same fraction $f < a$ of each species of insect. Show that the harmful population x_E is increased, while the benign population y_E is decreased, so the use of the insecticide is counterproductive. This is an instance in which mathematical analysis reveals undesirable consequences of a well-intentioned interference with nature.

More room for Problem 6.3.3.

For problems 6.3.4 through 6.3.7 we deal with the competition system

$$\frac{dx}{dt} = 60x - 4x^2 - 3xy,$$

$$\frac{dy}{dt} = 42y - 2y^2 - 3xy,$$

in which $c_1c_2 = 9 > 8 = b_1b_2$, so the effect of competition should exceed that of inhibition. Problems 6.3.4 through 6.3.7 imply that the four critical points $(0,0), (0,21), (15,0), (6,12)$ of the system resemble those showin in figure 6.3.9 from the textbook - a nodal source at the origin, a nodal sink at each coordinate axis, and a saddle point interior to the first quadrant. In each of these problems construct a phase portrait for the linearization at the indicated critical point. Finally, construct a phase portrait for the nonlinear system.

6.3.4 - Show that the coefficient matrix of the linearization

$$x' = 60x,$$

$$y' = 42y$$

of the system at $(0,0)$ has positive eigenvalues $\lambda_1 = 60$ and $\lambda_2 = 42$. Hence $(0,0)$ is a nodal source.

6.3.5 - Show that the linearizaion of the system at $(0, 21)$ is

$$u' = -3u,$$

$$v' = -63u - 42v.$$

Then show that the coefficient matrix of this linear system has negative eigenvalues $\lambda_1 = -3$ and $\lambda_2 = -42$. Hence $(0, 21)$ is a nodal sink for the system.

6.3.6 - Show that the linearization of the system at $(15, 0)$ is

$$u' = -60u - 45v,$$

$$4v' = -3v.$$

Then show that the coefficient matrix of this linear system has negative eigenvalues $\lambda_1 = -60$ and $\lambda_2 = -3$. Hence $(15, 0)$ is a nodal sink for the system.

6.3.7 - Show that the linearization of the system at $(6, 12)$ is

$$u' = -24u - 18v,$$

$$v' = -36u - 24v.$$

Then show that the coefficient matrix of this linear system has eigenvalues $\lambda_1 = -24 - 18\sqrt{2} < 0$ and $\lambda_2 = 24 + 18\sqrt{2} > 0$. Hence $(6, 12)$ is a saddle point for the system.

Draw the phase portrait for the system here.