Chapter 3

Explanations

3.1 Number and Operation

1. These collections of numbers are nested in each other. The integers are all whole numbers and their opposites (usually called negatives). The rational numbers are all the fractions and their opposites. The real numbers are all points on the number line, or what is the same, anything with a (possibly infinite) decimal expansion.

   So all of the given numbers are real numbers.

   (a) $\mathbb{Z}, \mathbb{Q}, \mathbb{R}$ – all the sets, because 3 is in fact a whole number.

   (b) $\mathbb{Q}, \mathbb{R}$ – $\frac{4}{3}$ can't be written as a whole number.

   (c) $\mathbb{Z}, \mathbb{Q}, \mathbb{R}$ – although presented as a fraction, it is a whole number.

   (d) $\mathbb{Q}, \mathbb{R}$ – 1.3 could be written $\dfrac{13}{10}$ – it is a rational. Any real number with a terminating decimal expansion is.

   (e) $\mathbb{Q}, \mathbb{R}$ – any real number with a repeating decimal expansion is also a rational. The bar over the 3 in this number indicates that the 3s go on forever.

   (f) $\mathbb{R}$ – $\sqrt{2}$ cannot be written as a fraction.

   (g) $\mathbb{Z}, \mathbb{Q}, \mathbb{R}$ – the ".0" at the end does not change that this number is just 4.

   (h) $\mathbb{R}$ – π has a non-terminating, non-repeating decimal expansion.

   (i) $\mathbb{Z}, \mathbb{Q}, \mathbb{R}$

   (j) $\mathbb{Q}, \mathbb{R}$

2. Two comments:

   C: How do you do a square root without a calculator? Don't worry too much about it. Just find the two closest perfect squares to the number under the root. 2 is between 1 and 4, which are 1^2 and 2^2. So $\sqrt{2}$ is between 1 and 2.

   D: Remember that -4.3 is *more negative* than -4, so it's farther left.

3. (a) A good strategy is to read the negative sign as "opposite". The opposite of the opposite is...

   (b) No comment.

   (c) Absolute value doesn't affect what is written outside it. We do it first, then take the opposite.

   (d) Similar to previous. Absolute value of -5 is 5. Then we take the opposite.

(e) If you got 12, you probably were thinking that absolute value "changes minus to plus". That's not exactly right. You need to evaluate everything inside the absolute value, then take the nonnegative form of the number left. Absolute value doesn't change operation signs inside itself.

4. (a) $\frac{2}{3}+\frac{4}{5}=\frac{10}{15}+\frac{12}{15}=\frac{22}{15}$. Usually in algebra we don't care about "improper fractions", when the denominator is larger than the numerator.

(b) $125-(-327)=125+327$. Subtracting the opposite of a number is the same as adding that number.

(c) This is how I would do it. First you need a common denominator

$$4\frac{3}{4}-3\frac{7}{8}=4\frac{6}{8}-3\frac{7}{8}$$

Now exchange a "1" from the 4 for 8 eighths:

$$=3\frac{14}{8}-3\frac{7}{8}$$

Now the 3s go, and

$$=\frac{14}{8}-\frac{7}{8}=\frac{7}{8}.$$

Here, changing to "improper fractions" also works fine, but is more work.

(d) If you got $12\frac{21}{32}$, be careful: you can't just multiply the whole number and fraction parts separately. Don't believe me? Try that on $1\frac{1}{2}\cdot 1\frac{1}{2}$ and see if your answer seems reasonable. Here, we better use improper fractions:

$$\begin{aligned} 4\frac{3}{4}\cdot 3\frac{7}{8} &= \frac{19}{4}\cdot\frac{31}{8} \\ &= \frac{19\cdot 31}{4\cdot 8} \\ &= \frac{589}{32} \end{aligned}$$

(e) Remember the "rule"? Multiply, ignoring the decimal points, count the number of places total after the decimal in the question, and place the decimal point that many digits from the right. It works, because

$$\begin{aligned} 1.5\cdot 2.25 &= \frac{15}{10}\cdot\frac{225}{100} \\ &= \frac{15\cdot 225}{10\cdot 100} \\ &= \frac{3375}{1000}=3.375 \end{aligned}$$

(f) This is equivalent to $22.5\div 15$.

(g) No comment.

(h) If you broke a debt of 91 dollars into groups of debts of 13 dollars, you'd have 7 groups.

5. (a) No comment.

(b) Negative exponents *don't* make the result negative. They mean to take the reciprocal before applying the corresponding positive power. This is done to make the rules of exponents consistent (ask me!). So

$$2^{-4} = \frac{1}{2^4} = \frac{1}{16}$$

(c) No comment.

(d) Fractional exponents correspond to roots: $9^{\frac{1}{2}} = \sqrt{9} = 3$.

(e) Combination of exponent rules.

(f) Exponents have higher precedence than the minus sign! So we raise 4 to the second power first, then take the opposite of that.

(g) Here the parentheses tell us to take the opposite of four and raise that to the second power.

6. General comment: Many people learn the order of operations by the mnemonic PEMDAS ("please excuse my dear Aunt Sally") – parentheses, exponentiation, multiplication, division, addition, subtraction. If you do this, remember that "MD" are really done together, left to right, when they occur. You don't do "M" before "D", just before "A" and "S". Same for "AS".

   (a) If you got 1, read what I wrote above.

   (b) If you got 1, read what I wrote above.

   (c) Parens tell you what to do before squaring.

   (d) Evaluate the whole numerator and whole denominator (with proper order of operations) first. Then reduce the resulting fraction. No cancelling until you've done all the addition and subtraction!

7. Distances are always positive. Subtract two numbers to find the distance between them. If you get a negative answer, change it positive.

3.2 Expressions

Mathematical expressions are fragments of mathematics with no mathematical "verb" in them (mathematical verbs are usually things like $=$, $<$, $>$, and things like that). An expression can have variables in it, but it cannot be solved for any of the variables, because an expression is not stating that anything is true.

Something like $3x+4$ is an expression. You can't find x because nothing is being stated about x. On the other hand, $3x+4 = 13$ is an equation. Something is being said to be true of x. When that is the case, our challenge is to find out what x must be. Stay tuned.

Remembering this statement takes care of a lot of mistakes involving expressions: If it's not set equal to zero, don't set it equal to zero.

Here are the things that we can do with expressions:

1. Simplify them: $3x+x$ becomes $4x$, $\frac{4x}{6}$ becomes $\frac{2x}{3}$.

2. Evaluate them at some value(s) of the variable(s): the expression $3x+4$ evaluates to 13 when x is set to 3. The expression $\frac{2x+3}{x-2}$ cannot be evaluated when $x = 2$ (why not?).

3. Build them from words. If I have n nickels and d dimes, a mathematical expression for the number of cents that I have is $5n+10d$.

3.3 Polynomial expressions

Certain expressions we meet a lot. Since we meet them a lot, there is a lot of terminology for them.
A *polynomial* is an expression of the form:

$$a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

where the a_n are real numbers, and n is a whole number.
Some terminology:

1. n is called the degree of the polynomial (note that n is the highest power occuring on an x if the terms happen not to be written in descending order, as here).
2. The numbers a_i are called *coefficients*.
3. The number a_n is called the *leading coefficient*.
4. The number a_0 is called the *constant term*.

When

- $n = 0$, the polynomial is called *constant*.
- $n = 1$, the polynomial is called *linear*.
- there is a total of one term, the polynomial is called a *monomial*.
- there is a total of two terms, the polynomial is called a *binomial*.
- there is a total of three terms, the polynomials is called a *trinomial*.

Once we enter the world of polynomial expressions, there is one more thing we can do with expressions:

- factor: write an expression as the product of two other expressions.

Factoring out the greatest monomial factor: here we try to find the factors that are common to every term in an expression, both those made of variables, and not.
Examples:

1. $2x^2 + 6x - 4 = 2(x^2 + 3x - 2)$
2. $x^3y^2 - x^2y + x^4y^3 = x^2y(xy - 1 + x^2y^2)$
3. $4x(x+2) - 3(x+2) = (x+2)(4x-3)$

This is the first thing to try when factoring anything.

3.3.1 Multiplying

1. $$(x-2)(5x-2) = 5x^2 - 2x - 10x + 4 = 5x^2 - 12x + 4)$$

2. $$\begin{aligned}(x-1)^2(x+2) &= (x^2 - 2x + 1)(x+2) \\ &= x^3 + 2x^2 - 2x^2 - 4x + x + 2 \\ &= x^3 - 3x + 2\end{aligned}$$

3.

$$\begin{aligned}(x-2)^3 &= (x^2-4x+4)(x-2)\\ &= x^3-2x^2-4x^2-8x+4x-8\\ &= x^3-6x^2-4x-8\end{aligned}$$

4.

$$3(x-2)(x+3)=3(x^2+3x-2x-6)=3(x^2+x-6)=3x^2+3x-18$$

5.

$$(5x+3)(x^2-3x-4)=5x^3+3x^2-15x^2-20x-9x-12=5x^3-12x^2-29x-12$$

6.

$$\begin{aligned}(9-6i)^2 &= (9-6i)(9-6i)\\ &= 81-54i-54i+36i^2\\ &= 81-108i-36\\ &= 45-108i\end{aligned}$$

3.3.2 Factoring

1. $144 = 16\cdot 9$. Factors are probably like $(x+?y)$...

$$x^2+7xy-144y^2=(x+16y)(x-9y)$$

2.

$$\begin{aligned}10x^2+6xy-25xy-15y^2 &= 2x(5x+3y)-5y(5x+3y)\\ &= (5x+3y)(2x-5y)\end{aligned}$$

3.

$$3x^3y^2-75xy^2=3xy^2(x^2-25)=3xy^2(x+5)(x-5)$$

4.

$$\begin{aligned}36x^3+12x^2-48x &= 12x(3x^2+x-4)\\ &= 12x(3x^2-3x+4x-4)\\ &= 12x[3x(x-1)+4(x-1)]\\ &= 12x(x-1)(3x+4)\end{aligned}$$

5.

$$\begin{aligned}x^3+3x^2-x-3 &= x^2(x+3)-1(x+3)\\ &= (x+3)(x^2-1)\\ &= (x+3)(x+1)(x-1)\end{aligned}$$

6.

$$\begin{aligned}x^4-y^4 &= (x^2)^2-(y^2)^2\\ &= (x^2+y^2)(x^2-y^2)\\ &= (x^2+y^2)(x+y)(x-y)\end{aligned}$$

3.4 Rational expressions

1.

$$\begin{aligned}\frac{2x^2+5x-3}{x^2+2x-3} &= \frac{2x^2+6x-x-3}{(x+3)(x-1)}\\ &= \frac{(2x-1)(x+3)}{(x+3)(x-1)}\\ &= \frac{2x-1}{x-1}, x \neq -3\end{aligned}$$

2.

$$\begin{aligned}\frac{-4x^3y^2}{2x^2y^3}\cdot\frac{-4xy^3}{10xy^2} &= \frac{-2x}{y}\cdot\frac{-2y}{5}\\ &= \frac{4xy}{5y} = \frac{4x}{5}, x,y \neq 0\end{aligned}$$

3.

$$\begin{aligned}\frac{3}{x-2}+\frac{3x}{2x+3} &= \frac{3(2x+3)}{(x-2)(2x+3)}+\frac{3x(x-2)}{(2x+3)(x-2)}\\ &= \frac{3(2x+3)+3x(x-2)}{(2x+3)(x-2)}\\ &= \frac{6x+9+3x^2-6x}{(2x+3)(x-2)}\\ &= \frac{3(x^2+3)}{(2x+3)(x-2)}\end{aligned}$$

4.

$$\begin{aligned}\frac{4x-48}{x^2-144} &= \frac{4(x-12)}{(x+12)(x-12)}\\ &= \frac{4}{x+12}, x \neq 12\end{aligned}$$

5.

$$\begin{aligned}\frac{x^2-9}{2x+2}\cdot\frac{x^2+2x+1}{(x-3)(x+1)} &= \frac{(x+3)(x-3)}{2(x+1)}\cdot\frac{(x+1)(x+1)}{(x-3)(x+1)}\\ &= \frac{x+3}{2}, x \neq -1,1,3\end{aligned}$$

6.

$$\begin{aligned}\frac{x^2+x-2}{3x^2-9x}\div\frac{x^2+4x-5}{6x-18}\cdot\frac{x^2+2x-15}{x^2-x-6} &= \frac{(x+2)(x-1)}{3x(x-3)}\cdot\frac{6(x-3)}{(x+5)(x-1)}\cdot\frac{(x-3)(x+5)}{(x-3)(x+2)}\\ &= \frac{2}{x}, x \neq -5,-2,1,3\end{aligned}$$

3.4.1 Complex fractions

1.

$$\begin{aligned}\frac{\frac{x+5}{3x^2}}{\frac{x^2-25}{6x^3}} &= \frac{x+5}{3x^2}\cdot\frac{6x^3}{x^2-25}\\ &= \frac{(x+5)\cdot 6x^3}{3x^2(x+5)(x-5)}\\ &= \frac{2x}{x-5}, x\neq 0,-5\end{aligned}$$

2. $\dfrac{4+\frac{1}{2}}{\frac{1}{3}+\frac{1}{6}} = \dfrac{\frac{9}{2}}{\frac{1}{2}} = 9$

3.

$$\begin{aligned}\frac{\frac{5}{x-5}+\frac{3}{x+3}}{\frac{5}{x+3}+\frac{3}{x-5}} &= \frac{\frac{5(x+3)}{(x-5)(x+3)}+\frac{3(x-5)}{(x+3)(x-5)}}{\frac{5(x-5)}{(x+3)(x-5)}+\frac{3(x+3)}{(x-5)(x+3)}}\\ &= \frac{\frac{5x+15+3x-15}{(x+3)(x-5)}}{\frac{5x-25+3x+9}{(x+3)(x-5)}}\\ &= \frac{8x}{8x-16} = \frac{8x}{8(x-2)} = \frac{x}{x-2}, x\neq -3,5\end{aligned}$$

3.5 Radical expressions

1.

$$\begin{aligned}3\sqrt{75}+4\sqrt{12} &= 3\sqrt{3\cdot 25}+4\sqrt{3\cdot 4}\\ &= 3\cdot 5\sqrt{3}+4\cdot 2\sqrt{3}\\ &= 15\sqrt{3}+8\sqrt{3} = 23\sqrt{3}\end{aligned}$$

2.

$$\begin{aligned}\left(\sqrt{14x^3y}\right)\left(\sqrt{7x^3y^3}\right) &= \sqrt{14\cdot 7x^6y^4}\\ &= \sqrt{2\cdot 7^2(x^3)^2(y^2)^2}\\ &= 7x^3y^2\sqrt{2}\end{aligned}$$

3.

$$\begin{aligned}\sqrt{\frac{375x^5}{5x}} &= \sqrt{75x^4}\\ &= \sqrt{3\cdot 25(x^2)^2}\\ &= 5x^2\sqrt{3}\end{aligned}$$

4.

$$\begin{aligned}\left(4\sqrt{5}-2\right)\left(2\sqrt{5}+4\right) &= 4\sqrt{5}\cdot 2\sqrt{5}+16\sqrt{5}-4\sqrt{5}-8\\ &= 8\cdot 5+12\sqrt{5}-8\\ &= 32+12\sqrt{5}\end{aligned}$$

5.

$$\begin{aligned}\sqrt[3]{xy^5}\sqrt[3]{x^7y^{17}} &= \sqrt[3]{x^8y^22}\\ &= \sqrt[3]{x^6x^2y^21y} = x^2y^7\sqrt[3]{x^2y}\end{aligned}$$

6.

$$\begin{aligned}\frac{\sqrt[5]{320x^{13}y^{18}}}{\sqrt[5]{10x^3y^3}} &= \sqrt[5]{\frac{320x^{13}y^{18}}{10x^3y^3}}\\ &= \sqrt[5]{32x^{10}y^{15}} = 2x^2y^3\end{aligned}$$

7.

$$\sqrt{16x^2+4} = \sqrt{4(4x^2+1)} = 2\sqrt{4x^2+1}$$

Careful here: square roots don't get along well with addition. I like to think that it's because they're two levels away in the order of operations.

1. The powers of i go in a cycle four members long. $i^1 = i$. $i^2 = -1$ by definition. So $i^3 = i\cdot i^2 = -i$, and then $i^4 = (i^2)^2 = 1$. So every power of i that is a multiple of four is equal to 1. For this problem: $i^{37} = i^{36}\cdot i = 1\cdot i = i$.

2.

$$(2+3i)(4-i) = 8-2i+12i-3i^2 = 8+10i+3 = 11+10i$$

3. To put a fraction like this into standard form, we need to make the denominator real. Any complex number multiplied by its *conjugate* (the complex number with the same real part but opposite imaginary part) yields a real number (try it!). A fraction is not changed by multiplying numerator and denominator by the same quantity, so

$$\begin{aligned}\frac{2+3i}{4-i} &= \frac{(2+3i)(4+i)}{(4-i)(4+i)}\\ &= \frac{8+2i+12i+3i^2}{16-4i+4i-i^2}\\ &= \frac{5+14i}{17}\\ &= \frac{5}{17}+\frac{14}{17}i\end{aligned}$$

3.6 Linear equations and inequalities

3.6.1 Equations

General tips: If two things are equal, and we do the same thing to both of them, they stay equal. This justifies "adding the same thing to both sides", and so on.

A good strategy, when trying to get all the expressions with x on the same side is to move so that the coefficient of x is positive. Fewer arithmetic errors are made when dividing by positive numbers.

1.

$$\begin{aligned} 3x+4=6-x &\Rightarrow \quad 4x=2 \\ &\Rightarrow \quad x=\frac{1}{2} \end{aligned}$$

2.

$$\begin{aligned} -3x+4=6-x &\Rightarrow \quad -2=2x \\ &\Rightarrow \quad x=-1 \end{aligned}$$

3.

$$\begin{aligned} \frac{3}{x}=\frac{9}{14} &\Rightarrow \quad 9x=3\cdot 14 \\ &\Rightarrow \quad x=\frac{3\cdot 14}{9}=\frac{14}{3} \end{aligned}$$

4.

$$\begin{aligned} \frac{x}{7}=\frac{3}{5} &\Rightarrow \quad 5x=3\cdot 7 \\ &\Rightarrow \quad x=\frac{21}{5} \end{aligned}$$

5.

$$\begin{aligned} \frac{3+x}{5}=\frac{7}{10} &\Rightarrow \quad 10(3+x)=5\cdot 7 \\ &\Rightarrow \quad 30+10x=35 \\ &\Rightarrow \quad 10x=5 \\ &\Rightarrow \quad x=\frac{1}{2} \end{aligned}$$

6.

$$\begin{aligned} \frac{3-x}{x}=\frac{3}{5} &\Rightarrow \quad 5(3-x)=3x \\ &\Rightarrow \quad 15-5x=3x \\ &\Rightarrow \quad 15=8x \\ &\Rightarrow \quad x=\frac{8}{15} \end{aligned}$$

7.

$$\begin{aligned} \frac{x-2}{4}=\frac{x+1}{12} &\Rightarrow \quad 12(x-2)=4(x+1) \\ &\Rightarrow \quad 12x-24=4x+4 \\ &\Rightarrow \quad 8x=28 \\ &\Rightarrow \quad x=\frac{28}{8}=\frac{7}{2} \end{aligned}$$

3.6.2 Harder Equations

1.

$$\begin{aligned}\frac{x}{3}-2=\frac{3}{10} &\Rightarrow 3\cdot 10\cdot\left(\frac{x}{3}-2\right)=3\cdot 10\cdot\frac{3}{10}\\ &\Rightarrow 3\cdot 10\cdot\frac{x}{3}-3\cdot 10\cdot 2=3\cdot 3\\ &\Rightarrow 10x-60=9\\ &\Rightarrow 10x=69\\ &\Rightarrow x=\frac{69}{10}\end{aligned}$$

2.

$$\begin{aligned}\frac{2x-3}{5}+x=\frac{4-x}{4}+3 &\Rightarrow 5\cdot 4\cdot\left(\frac{2x-3}{5}+x\right)=5\cdot 4\cdot\left(\frac{4-x}{4}+3\right)\\ &\Rightarrow 5\cdot 4\cdot\frac{2x-3}{5}+5\cdot 4\cdot x=5\cdot 4\cdot\frac{4-x}{4}+5\cdot 4\cdot 3\\ &\Rightarrow 4(2x-3)+20x=5(4-x)+60\\ &\Rightarrow 8x-12+20x=20-5x+60\\ &\Rightarrow 28x-12=-5x+80\\ &\Rightarrow 33x=92\\ &\Rightarrow x=\frac{92}{33}\end{aligned}$$

3.

$$\begin{aligned}\frac{2}{x-1}=\frac{8}{3x+6} &\Rightarrow (x-1)(3x+6)\frac{2}{x-1}=(x-1)(3x+6)\frac{8}{3x+6}\\ &\Rightarrow (3x+6)\cdot 2=(x-1)\cdot 8\\ &\Rightarrow 6x+12=8x-8\\ &\Rightarrow 20=2x\\ &\Rightarrow x=10\end{aligned}$$

4.

$$\begin{aligned}\frac{3x+3}{3}=\frac{2x+2}{2} &\Rightarrow 2(3x+3)=3(2x+2)\\ &\Rightarrow 6x+6=6x+6\end{aligned}$$

Those two things are the same no matter what you choose for x!

5.

$$\begin{aligned}\frac{3x+3}{3}=\frac{2x+3}{2} &\Rightarrow 2(3x+3)=3(2x+3)\\ &\Rightarrow 6x+6=6x+9\\ &\Rightarrow 6=9\end{aligned}$$

That's never true, no matter what x is! (x actually left the equation!)

6.

$$\begin{aligned}\frac{1}{x+3}-2=5 &\Rightarrow (x+3)\left(\frac{1}{x+3}-2\right)=(x+3)\cdot 5\\ &\Rightarrow (x+3)\frac{1}{x+3}-(x+3)\cdot 2=5x+15\\ &\Rightarrow 1-2x-6=5x+15\\ &\Rightarrow -2x-5=5x+15\\ &\Rightarrow -20=7x\\ &\Rightarrow x=-\frac{7}{20}\end{aligned}$$

3.6.3 Inequalities

1.

$$\begin{aligned}3x+4<6-x &\Rightarrow 4x<2\\ &\Rightarrow x<\frac{1}{2}.x\in(-\infty,\frac{1}{2})\end{aligned}$$

2.

$$\begin{aligned}-3x+4\geq 6-x &\Rightarrow -2\geq 2x\\ &\Rightarrow -1\geq x.x\in[-1,\infty)\end{aligned}$$

3.

$$\begin{aligned}\frac{x}{7}<\frac{3}{5} &\Rightarrow 5x<21\\ &\Rightarrow x<\frac{21}{5}.x\in(-\infty,\frac{21}{5})\end{aligned}$$

4.

$$\begin{aligned}\frac{3+x}{5}\leq\frac{7}{10} &\Rightarrow 10(3+x)\leq 5\cdot 7\\ &\Rightarrow 30+10x\leq 35\\ &\Rightarrow 10x\leq 5\\ &\Rightarrow x\leq\frac{1}{2}.x\in(-\infty,\frac{1}{2}]\end{aligned}$$

5.

$$\begin{aligned}\frac{x-2}{4}\geq\frac{x+1}{12} &\Rightarrow 12(x-2)\geq 4(x+1)\\ &\Rightarrow 12x-24\geq 4x+4\\ &\Rightarrow 8x\geq 28\\ &\Rightarrow x\geq\frac{28}{8}.x\in[\frac{7}{2},\infty)\end{aligned}$$

6.

$$\begin{aligned}-3<2x+5\leq 7 &\Rightarrow -8<2x\leq 2\\ &\Rightarrow -4<x\leq 1.x\in(-4,1]\end{aligned}$$

3.7 Systems of linear equations

In this section, the problems have multiple variables. Since the solution to such a system would have an equality for each of the variables, if we expect a unique solution, we need to start with as many equations as variables. Sometimes we are fortunate, and one of the given equations gives the value of one of the variables. Usually, we are not. In this case, since both (or all) of the given equations are assumed to be true, the equations may be manipulated separately (both sides multiplied by constants), or combined (the equations may be added to or subtracted from each other). The goal is to combine them in such a way that an equation with only one variable results. Once you know one of the variables, you back to what's left and try to get the rest.

1. In this system, we are given the value of y right away:

$$\begin{cases} 2x + 3y = 7 \\ y = 4 \end{cases}$$

Substituting $y = 4$ in the first equation gives:

$$2x + 3(4) = 7$$

Solving this gives $x = -\frac{5}{2}$.

2. Decimals shouldn't be scary. If they are, just pre-multiply both sides of both equations by ten. That would turn this into a system with all whole number coefficients. We'll work it with the decimals in.

$$\begin{cases} 0.4x + 0.6y = 3.2 \\ 1.4x - 0.3y = 1.6 \end{cases}$$

Multiplying the second equation by 2 gives:

$$2.8x - 0.6y = 3.2$$

Adding this to the original first equation will eliminate y:

$$3.2x = 6.4$$

This implies $x = 2$. We can substitute this into any previous equation. Let's use the first one:

$$0.4(2) + 0.6y = 3.2$$

A couple of steps gives $y = 4$.

3.

$$\begin{cases} 2x - 3y = 5 \\ 6x - 9y = 4 \end{cases}$$

Multiply the first equation by 3 to get:

$$6x - 9y = 15$$

But this *contradicts* the original second equation, which says that $6x - 9y$ equals something totally different: 4. So these equations can't be solved together. No solutions.

4.

$$\begin{cases} 2x - 3y = 5 \\ -6x + 9y = -15 \end{cases}$$

Similar start: multiply the first equation by -3 to get:

$$-6x+9y=-15$$

Now that's the *same* as the second equation. Since the two equations are deep down the same (one is just a multiple of the other), we have no hope of getting unique values for x and y. But that doesn't mean there is *no* solution. It means that there are infinitely many. Solve the first equation for y to get:

$$y=\frac{2}{3}x-\frac{5}{3}$$

Now since the equations are basically the same, you give me an x, say $x=3$, and I use the formula to give you the y that goes with it:

$$y=\frac{2}{3}(3)-\frac{5}{3}(3)=-3$$

But there was nothing special about $x=3$. You could give me any x. So there are infinitely many solutions, and we know just how to make them.

5. Here is just a longer back-substituting process:

$$\begin{cases} 2x - 3y + z = 5 \\ \quad\quad 2y + z = 4 \\ \quad\quad\quad\quad z = -2 \end{cases}$$

Plug the given z-value into the second equation:

$$2y+(-2)=4$$

That gives $y=3$. Now plug x and y values into the first equation:

$$2x-3(3)+(-2)=5$$

This gives $x=4$.

6.

$$\begin{cases} 2x - 3y + z = 1 \\ x + 2y + z = -1 \\ 3x - y + 3z = 4 \end{cases}$$

This kind of thing is really best handled by computer. But so you know, the strategy goes: eliminate a variable by combining the first two equations, then eliminate the same variable by combining the second two equations. Now we will have two equations in two variables, and may proceed as before.

Here, let's eliminate z. Subtracting equation 2 from equation 1:

$$x-5y=2$$

We triple equation 2 to get $3x+6y+3x=-1$. We subtract the third equation from this to get:

$$7y=-7$$

We were very fortunate. In general, we still have x and y, but here we know immediately that $y=-1$.

Something important: every intermediate equation that we've gotten is true! So we can use $x-5y=2$ with the fact $y=-1$ to get:

$$x-5(-1)=2\Rightarrow x=-3$$

Now pick any equation with a z in it, and use $x=-3$, $y=-1$:

$$2(-3)-3(-1)+z=1\Rightarrow -6+3+z=1\Rightarrow z=4$$

3.8 Absolute Value Equations and Inequalities

3.8.1 Equations

There isn't an operation that can just "undo" an absolute value in an equation. So the basic strategy is:

1. Solve the equation for the entire absolute value expression.
2. Now the stuff inside the absolute value can be either equal to the other side of the equation, or to its opposite. This means there may be multiple values of x. Note: in this step, if you've solved for the absolute value, and it equals a negative, the original equation has NO solutions. An absolute value cannot be negative.
3. If the equation from the previous step does have solutions, solve the resulting equations separately.

1.

$$\begin{aligned} |3x-2| = 6 &\Rightarrow 3x-2=6 \text{ or } 3x-2=-6 \\ &\Rightarrow 3x=8 \text{ or } 3x=-4 \\ &\Rightarrow x=\frac{8}{3} \text{ or } x=\frac{-4}{3} \end{aligned}$$

2.

$$\begin{aligned} \left|\frac{3-x}{3}\right|-4=-2 &\Rightarrow \\ &\Rightarrow \left|\frac{3-x}{3}\right|=2 \\ &\Rightarrow \frac{3-x}{3}=2 \text{ or } \frac{3-x}{3}=-2 \\ &\Rightarrow 3-x=6 \text{ or } 3-x=-6 \\ &\Rightarrow x=-3 \text{ or } x=9 \end{aligned}$$

3.

$$|x+5|+7=5 \Rightarrow |x+5|=-2$$

But the $|x+5|$ is *never* negative. So there are no solutions.

4.

$$\begin{aligned} 2-\left|\frac{x}{5}-1\right|=1 &\Rightarrow \left|\frac{x}{5}-1\right|=1 \\ &\Rightarrow \frac{x}{5}-1=1 \text{ or } \frac{x}{5}-1=-1 \\ &\Rightarrow \frac{x}{5}=2 \text{ or } \frac{x}{5}=0 \\ &\Rightarrow x=10 \text{ or } x=0 \end{aligned}$$

5. This one is a little different. The equation says that two expressions have the same absolute value. This is possible if the expressions themselves are the same, or if they are opposite.

$$\begin{aligned} |2x+3|=|3x-2| &\Rightarrow 2x+3=3x-2 \text{ or } 2x+3=-(3x-2) \\ &\Rightarrow 5=x \text{ or } 2x+3=-3x+2 \\ &\Rightarrow x=5 \text{ or } 5x=-1 \\ &\Rightarrow x=5 \text{ or } x=\frac{-1}{5} \end{aligned}$$

3.8.2 Inequalities

Same basic principle: isolate any absolute value first.
If an absolute value is *less than* a given (positive!) number, that means that the expression in the absolute value is "not too positive, not too negative". The solution set is a single interval.
Think: How can the absolute value of a number be less than 5? Well, the number could be positive itself, and less than five. Or, if the number is negative, it can't be more negative than -5 (that is, greater than -5).
If an absolute value is *greater than* a given (positive!) number, that means that the expression in the absolute value is "away from zero, left or right". The solution set is the union of two intervals.
Think: How can the absolute value of a number be greater than 5? If the number is positive, it must itself be greater than five. If it's negative, it must be more negative than negative five (that is, less than -5).

1.

$$\begin{aligned} |3x-4| < 6 &\Rightarrow -6 < 3x-4 < 6 \\ &\Rightarrow -2 < 3x < 10 \\ &\Rightarrow \frac{-2}{3} < x < \frac{10}{3} \\ &\Rightarrow x \in (-2/3, 10/3) \end{aligned}$$

2.

$$\begin{aligned} |4-x| \geq 3 &\Rightarrow 4-x \geq 3 \text{ or } 4-x \leq -3 \\ &\Rightarrow 1 \geq x \text{ or } 7 \leq x \\ &\Rightarrow x \in (-\infty, 1] \cup [7, \infty) \end{aligned}$$

3.

$$\begin{aligned} |2-3x| \leq 6 &\Rightarrow -6 \leq 2-3x \leq 6 \\ &\Rightarrow -8 \leq -3x \leq 4 \\ &\Rightarrow \frac{8}{3} \geq x \geq \frac{-4}{3} \\ &\Rightarrow x \in \left[\frac{-4}{3}, \frac{8}{3}\right] \end{aligned}$$

4.

$$|-x-5|+5 \leq 3 \Rightarrow |-x-5| \leq -2$$

This is impossible, because $|-x-5|$ cannot be negative.

5. Any number has an absolute value greater than -3!

6. This one is tricky. The absolute value is trapped between 2 and 4, so the expression inside is either itself between 2 and 4, or between -4 and -2.

$$\begin{aligned} 2 < \left|\frac{x+2}{3} - 1\right| < 4 &\Rightarrow 2 < \frac{x+2}{3} - 1 < 4 \text{ or } -4 < \frac{x+2}{3} - 1 < 2 \\ &\Rightarrow 3 < \frac{x+2}{3} < 5 \text{ or } -3 < \frac{x+2}{3} < -1 \\ &\Rightarrow 9 < x+2 < 15 \text{ or } -9 < x+2 < -3 \\ &\Rightarrow 7 < x < 13 \text{ or } -11 < x < -5 \\ &\Rightarrow x \in (-11, -5) \cup (7, 13) \end{aligned}$$

3.9 Functions and graphs

3.9.1 Functions

1. Yes. This is already solved for y, so a value of x uniquely determines y.
2. No. If we try to solve for y, we end up taking a square root, which could have positive or negative values. So y is not uniquely determined.
3. Yes. Odd degree roots are unique.
4. Yes. Here we can solve for y. We square both sides at some point, but that does not cause a problem.
5. Yes. Each x value appears no more than once.
6. Yes. Each x value appears no more than once.
7. No. x values appear multiple times.
8. No. Fails the vertical line test.
9. Yes. Passes the vertical line test.

3.10 Graphs of linear equations and inequalities

3.10.1 Drawing graphs

1. $y = 2x+3$: This one is in slope-intercept form. Plot the $y-$intercept $(0,3)$, then count "two up, one right" to get another point, and connect the points with a line.
2. $y = -\frac{1}{2}x-2$: Same idea. Negative slope here: "one down, two right".
3. $y = 2-3x$: Don't be fooled by the terms written in non-standard order. The y-intercept here is positive, and the slope is negative.
4. $2x+3y = 6$: You can always solve for y and plot from slope-intercept form. But I prefer to find two points. Set $x = 0$ and solve for y: plot $(0,2)$. Set $y = 0$ and solve for x: plot $(3,0)$. Connect.
5. $x = 4$: All the points with $x = 4$ make up a vertical line.
6. $0.2x+0.6y = 1.2$: This is easier to plot if we first multiply both sides by ten: $2x+6y = 12$. Set $x = 0$ to find the y-intercept, then $y = 0$ to find the x-intercept...
7. $y < -\frac{1}{2}x+4$: y is on the "less than" side, so we shade below. the line is dotted, since the inequality is strict.
8. $2x-4y \geq 3$: Inequality is not strict, so line is solid. You can solve this for y and plot, or find the intercepts $(1.5,0)$, $(0,-0.75)$ and plot the line. How do you know which way to shade? Test with $(0,0)$. Is it true that $2(0)-4(0) \geq 3$. No. So shade on the other side of the line (the one away from $(0,0)$).
9. $x > 2y-5$: x-intercept is $(-5,0)$. y-intercept is $(0,2.5)$. Plot the (dotted) line. Check $(0,0)$. Is $0 > 2(0)-5$? Yes. So shad the region containing $(0,0)$.

3.10.2 Finding equations of lines

Find the equations of lines (in slope-intercept form)

1. slope=-2, y-intercept (0,-1). No problem; that's what slope-intercept form is made of: $y = -2x - 1$.

2. Find the slope first: change in y over change in x.

$$m = \frac{3-(-2)}{-1-2} = \frac{5}{-3} = -\frac{5}{3}$$

   So the equation we want must look like:

$$y = -\frac{5}{3}x + b$$

   But if $(-1,3)$ is on the line, then

$$3 = -\frac{5}{3}(-1) + b$$

   Solve for b: $b = \frac{1}{3}$.

3. through the points (1,1) and (1,-5). The two given points have the same x-coordinate. The only way that a line can do that is if all the other points also have the same x-coordinate. So the equation is $x = 1$. Note that if you tried to find the slope, you'd be dividing by zero. So such a line is said to have no slope (not zero slope!).

4. with y-intercept $(0,3)$ and parallel to $y = 2x - 4$. Parallel lines have the same slope, so our line will have slope 2 as well:

$$y = 2x + b$$

   But we were given the y-intecept, so $y = 2x + 3$.

5. through (-1,1) and perpendicular to $2x + y = 5$. The given line in slope-intercept form is:

$$y = -2x + 5$$

   That line has slope -2. A perpendicular line has a slope that is the opposite reciprocal of this, that is, $\frac{1}{2}$. So our line is like

$$y = \frac{1}{2}x + b$$

   But if the line goes through $(-1,1)$,

$$1 = \frac{1}{2}(-1) + b$$

   So our line is $y = \frac{1}{2}x + \frac{3}{2}$.

6. through (2,-3) and perpendicular to $y = 4$. A line perpendicular to a horizontal line like $y = 4$ is vertical (all x-coordinates the same). Since the given point has $x = 2$, the line's equation must also be simply $x = 2$.

3.11 Polynomial equations

When the same variable occurs in the same equation to different powers, there are basically no valid algebraic operations that will allow you to isolate the variable. As an alternative, in this case, we move all the terms on to on side of the equation, leaving the other side zero. If the resulting expression is factorable, we can use the *zero property of multiplication*: if a number of factors are multiplied and the result is zero, then one of those factors had to be zero itself.
Some polynomial equations do not require this. Using some clever manipulations (one of which is *completing the square*), these equations can be solved by taking roots.

1.
$$x^2-2x+1=0 \Rightarrow (x-1)(x-1)=0 \Rightarrow x=1$$

2.
$$x^2-8x+7=0 \Rightarrow (x-7)(x-1)=0 \Rightarrow x=1,7$$

3.
$$x^2-5x=6 \Rightarrow x^{-}5x-6=0 \Rightarrow (x-6)(x+1)=0 \Rightarrow x=-1,6$$

4.
$$x^2-2x=0 \Rightarrow x(x-2)=0 \Rightarrow x=0,2$$

5.
$$x^2-\frac{1}{4}=0 \Rightarrow x^2=\frac{1}{4} \Rightarrow x=\pm\sqrt{\frac{1}{4}} \Rightarrow x=\pm\frac{1}{2}$$

6.
$$x^2+16=0 \Rightarrow x^2=-16$$

No real number squares to a negative. This equation has complex solutions $x=\pm 4i$.

7.
$$\begin{aligned} 3x^2+6x=24 &\Rightarrow 3x^2+6x-24=0 \\ &\Rightarrow 3(x^2+2x-8)=0 \\ &\Rightarrow 3(x+4)(x-2)=0 \\ &\Rightarrow x=-4,2 \end{aligned}$$

8.
$$\begin{aligned} 2x^2-3x-2=0 &\Rightarrow 2x^2-4x+x-2=0 \\ &\Rightarrow 2x(x-2)+1(x-2)=0 \\ &\Rightarrow (x-2)(2x+1)=0 \\ &\Rightarrow x=-\frac{1}{2},2 \end{aligned}$$

9.
$$\begin{aligned} 6x^2=7x+3 &\Rightarrow 6x^2-7x-3=0 \\ &\Rightarrow 6x^2-9x+2x-3=0 \\ &\Rightarrow 3x(2x-3)+1(2x-3)=0 \\ &\Rightarrow (2x-3)(3x+1)=0 \\ &\Rightarrow x=\frac{3}{2},-\frac{1}{3} \end{aligned}$$

10.

$$\begin{aligned} 12x^2-13x+3=0 &\Rightarrow 12x^2-9x-4x+3=0 \\ &\Rightarrow 3x(4x-3)-1(4x-3)=0 \\ &\Rightarrow (4x-3)(3x-1)=0 \\ &\Rightarrow x=\frac{3}{4},\frac{1}{3} \end{aligned}$$

11.

$$\begin{aligned} x^4-2x^2+1=0 &\Rightarrow (x^2)^2-2(x^2)+1=0 \\ &\Rightarrow (x^2-1)^2=0 \\ &\Rightarrow x^2-1=0 \\ &\Rightarrow x=\pm 1 \end{aligned}$$

12.

$$\begin{aligned} x-8\sqrt{x}+7=0 &\Rightarrow (\sqrt{x})^2-8(\sqrt{x})+7=0 \\ &\Rightarrow (\sqrt{x}-1)(\sqrt{x}-7)=0 \\ &\Rightarrow \sqrt{x}=1 \text{ or } \sqrt{x}=7 \\ &\Rightarrow x=1,49 \end{aligned}$$

13.

$$\begin{aligned} x^3-5x^2=6x &\Rightarrow x^3-5x^2-6x=0 \\ &\Rightarrow x(x^2-5x-6)=0 \\ &\Rightarrow x(x-6)(x+1)=0 \\ &\Rightarrow x=-1,0,6 \end{aligned}$$

14.

$$x^4-4x^2=0 \Rightarrow x^2(x^2-4)=0 \Rightarrow x=-2,0,2$$

15.

$$(x-2)^2=7 \Rightarrow x-2=\pm\sqrt{7} \Rightarrow x=2\pm\sqrt{7}$$

16. Think: $x^2-2x+1=(x-1)^2$

$$\begin{aligned} x^2-2x-5=0 &\Rightarrow (x-1)^2-6=0 \\ &\Rightarrow (x-1)^2=6 \\ &\Rightarrow x-1=\pm\sqrt{6} \\ &\Rightarrow x=1\pm\sqrt{6} \end{aligned}$$

17. Quadratic formula

$$\begin{aligned} x^2+7x+3=0 &\Rightarrow x=\frac{-7\pm\sqrt{7^2-4\cdot 1\cdot 3}}{2\cdot 1} \\ &\Rightarrow x=\frac{-7\pm\sqrt{37}}{2} \end{aligned}$$

18. For $x^2+2x+3=0$, the discriminant $2^2-4\cdot 1\cdot 3=-8<0$, so the quadratic equation gives no real solutions.

19.
$$x^2+14x+49=0 \Rightarrow (x+7)^2=0 \Rightarrow x=-7$$

20.
$$x^2-14x+49=0 \Rightarrow (x-7)^2=0 \Rightarrow x=7$$

3.12 Graphs of quadratic functions

1. This is just "multiplying out":
$$y=2(x-1)^2+4=2(x^2-2x+1)+4=2x^2-4x+2+4=2x^2-4x+6$$

2. We can ignore the last term as a start. The first two terms are $2x^2+4x=2(x^2+2x)$. The perfect square closest to the expression in parentheses is $x^2+2x+1=(x-1)^2$. If we were to replace that, we would have: $2(x-1)^2=2x^2+4x+2$. But what we have is $2x^2+4x+5$, which is three more. So:
$$y=2x^2+4x+5=2(x-1)^2+3$$

3. The vertex of the standard parabola $y=x^2$ is $(0,0)$. This parabola is shifted right by 1 and up by 4. So its vertex is at $(1,4)$.

4. We wrote this parabola in vertex form above. So the vertex is $(-1,3)$.

5. The vertex form will look like $y=a(x+1)^2+3$ if the vertex is at $(-1,3)$. If the y-intercept is $(0,-2)$, then the equation at that point reads:
$$-2=a(0+1)^2+3 \Rightarrow -2=a+3$$
So $a=-5$, and the equation is $y=-5(x+1)^2+3$.

6. If a polynomial has a root of $x=a$, it has a factor of $(x-a)$. So a parabola with these roots could have equation $y=(x+2)(x-3)$. Multiplying this by a constant is not going to change where it equals zero, so any multiple of this would be fine as well.

7. See the answers for the graph. General strategy for graphing in vertex form is to plot the vertex. Then plug in $x=0$ to get the y-intercept and plot that. Reflect that point over the axis of symmetry (the vertical line through the vertex) to get a third point on the graph. Then draw the best parabola you can through those three points.

8. We have already done the work to put this in vertex form, so using that as in the previous example is the best bet.

3.13 Equations involving radicals

1.
$$\begin{aligned} 3\sqrt{5-2x}=9 &\Rightarrow \sqrt{5-2x}=3 \\ &\Rightarrow 5-2x=9 \\ &\Rightarrow -4=2x \\ &\Rightarrow x=-2 \end{aligned}$$

2.

$$\begin{aligned}\sqrt{2x+1}+1=4 &\Rightarrow \sqrt{2x+1}=3\\ &\Rightarrow 2x+1=9\\ &\Rightarrow x=4\end{aligned}$$

3.

$$\begin{aligned}y-2=\sqrt{y+4} &\Rightarrow (y-2)^2=y+4\\ &\Rightarrow y^2-4y+4=y+4\\ &\Rightarrow y^2-5y=0\\ &\Rightarrow y(y-5)=0\\ &\Rightarrow y=0 \text{ or } y=5\end{aligned}$$

But, if you check, $y=0$ does not work. The only solution is $y=5$.

And the difficult ones:

1.

$$\begin{aligned}\sqrt{5t}=1+\sqrt{5(t-1)} &\Rightarrow \sqrt{5t}-1=\sqrt{5(t-1)}\\ &\Rightarrow \left(\sqrt{5t}-1\right)^2=5(t-1)\\ &\Rightarrow 5t-2\sqrt{5t}+1=5t-5\\ &\Rightarrow 2\sqrt{5t}=6\\ &\Rightarrow 5t=9\Rightarrow t=\frac{9}{5}\end{aligned}$$

2.

$$\begin{aligned}\sqrt{1+6x}=2-\sqrt{6x} &\Rightarrow 1+6x=4-4\sqrt{6x}+6x\\ &\Rightarrow 4\sqrt{6x}=3\\ &\Rightarrow 16\cdot 6x=9\Rightarrow x=\frac{3}{32}\end{aligned}$$

3.14 Exponential and Logarithmic functions

To start: the rules of exponents with the corresponding rules of logarithms:

1. $a^m\cdot a^n=a^{m+n}$
2. $a^m/a^n=a^{m-n}$
3. $(a^m)^n=a^{mn}$
4. $a^0=1$, if $a\neq 0$.

1. $\log_a mn=\log_a m+\log_a n$
2. $\log_a(m/n)=\log_a m-\log_a n$
3. $\log_a x^m=m\log_a x$
4. $\log_a 1=0$

1.

$$8^{2/3}=(8^{1/3})^2=(\sqrt[3]{8})^2=2^2=4$$

2. Negative exponents indicate taking a reciprocal. They do not make the answer negative!

$$8^{-2/3}=\frac{1}{8^{2/3}}=\frac{1}{4}$$

3.
$$(-8)^{2/3} = [(-8)^{1/3}]^2 = (-2)^2 = 4$$

4.
$$(-8)^{-2/3} = \frac{1}{(-8)^{2/3}} = \frac{1}{4}$$

5. Order of operations here: exponents take precedence over the negative sign.
$$-8^{2/3} = -4$$

6. Powers are applied to numerator and denominator:
$$\left(\frac{4}{9}\right)^{3/2} = \frac{4^{3/2}}{9^{3/2}} = \frac{\sqrt{4}^3}{\sqrt{9}^3} = \frac{2^3}{3^3} = \frac{8}{27}$$

7. This logarithm asks the question, "what power of 2 is $\frac{1}{2}$?" Since $\frac{1}{2}$ is the reciprocal of 2, the answer is -1.

8. This one asks, "what power of 3 is 27?" The answer is 3.

9. What power of 9 is 3? Since 3 is the square root of 9, the answer is $\frac{1}{2}$.

1. Rewrite this one in exponential form. When log has no shown base, the base is ten:
$$x = 10^{-3} = 0.001$$

2.
$$x = 2^{-1/2} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

3.
$$\begin{aligned} e^{2x} - 2 = 0 \quad &\Rightarrow \quad e^{2x} = 2 \\ &\Rightarrow \quad 2x = \ln 2 \\ &\Rightarrow \quad x = \frac{1}{2}\ln 2 \end{aligned}$$

4.
$$\begin{aligned} 8 - 3 \cdot 2^{0.5x} = -40 \quad &\Rightarrow \quad -3 \cdot 2^{0.5x} = -48 \\ &\Rightarrow \quad 2^{0.5x} = 16 \\ &\Rightarrow \quad 0.5x = 4 \Rightarrow x = 8 \end{aligned}$$

5.
$$\log_3 9x = 3 \Rightarrow 9x = 3^3 \Rightarrow 9x = 27 \Rightarrow x = 3$$

6. x occurs twice in this problem in places that cannot be combined. So we must find a way to factor:
$$e^{2x} - 3e^x = 28 \Rightarrow (e^x)^2 - 3(e^x) - 28 = 0$$
The left side looks like $y^2 - 3y - 28$, which factors as $(y-7)(y+4)$, but with y replaced by e^x. So:
$$(e^x - 7)(e^x + 4) = 0,$$
which means that $e^x = 7$ or $e^x = -4$. Since e^x is never negative, the only possiblity is $e^x = 7$, which means $x = \ln 7$.

7. We want to use rules of logarithms to combine the terms on the left:

$$\log(x+1) - \log(x-1) = \log 3 \Rightarrow \log \frac{x+1}{x-1} = \log 3$$

Now we have an equality of two logarithms. Since the logarithm function is one-to-one, the arguments of the logarithms must be equal:

$$\frac{x+1}{x-1} = 3 \Rightarrow x+1 = 3(x-1) \Rightarrow x+1 = 3x-3 \Rightarrow 4 = 2x \Rightarrow x = 2$$

Note that any time we solve a logarithm equation, we must check that the solution can be plugged in to the original equation. In this case, $x = 2$ is fine. If plugging it in made the argument of a logarithm negative, we have a false solution, which must be discarded.

8.

$$\begin{aligned} \ln(x^2-4) - \ln(x+2) = \ln(3-x) \quad &\Rightarrow \quad \ln \frac{x^2-4}{x+2} = \ln(3-x) \\ &\Rightarrow \quad \frac{x^2-4}{x+2} = 3-x \\ &\Rightarrow \quad \frac{(x+2)(x-2)}{x+2} = 3-x \\ &\Rightarrow \quad x-2 = 3-x \Rightarrow x = \frac{5}{2} \end{aligned}$$

And $x = \frac{5}{2}$ does not make any logarithm arguments in the original equation negative (nor any denominators zero!). So it is a valid solution.

Question: why combine logs? It makes equation solving easier.

1. Note here that the log terms cannot be combined until the coefficient $\frac{1}{2}$ has been brought inside the first as a power. In a sense, the rules of logarithms should be applied in the same order as order of operations (the $\frac{1}{2}$ becomes an exponent, so it's first).

$$\frac{1}{2}\log_2 x + \log_2 y = \log_2 x^{1/2} + \log_2 y = \log_2 \sqrt{x} + \log_2 y = \log_2 y\sqrt{x}$$

2. Same thing here:

$$\begin{aligned} 3\log x - 2\log y - \log z &= \log x^3 - \log y^2 - \log z \\ &= \log x^3 - (\log y^2 + \log z) \\ &= \log x^3 - \log y^2 z \\ &= \log \frac{x^3}{y^2 z} \end{aligned}$$

Why did I factor out the negative sign in the second line? Remember that order of operations for addition and subtraction goes right to left. So I would have to do two divisions if I didn't factor. This makes more room for error.

Question: why do we even learn about separating out logs? Answer: You need to do it in calculus.

1.

$$\log\left(\frac{x\sqrt{y}}{z}\right) = \log x\sqrt{y} - \log z = \log x + \log\sqrt{y} - \log z = \log x + \frac{1}{2}\log y - \log z$$

2.

$$\log_4 4x^2y^3 = \log_4 4 + \log_4 x^2 + \log_4 y^3 = 1 + 2\log_4 x + 3\log_4 y$$