Math 2270 - Computer Lab 4 : Fourier Series

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This lab should be a pretty quick lab. It's goal is to introduce you to one of the coolest ideas in mathematics, the Fourier series, and have you play around with them a little bit. We won't see much more about Fourier series in this class, but you should at least see them once.

The Idea of Fourier Series

Let $\mathbf{v} =< v_1, v_2, v_3 >$ be a vector in $\mathbb{R}^3$. The vectors $\hat{\mathbf{i}} =< 1, 0, 0 >, \hat{\mathbf{j}} =< 0, 1, 0 >, \hat{\mathbf{k}} =< 0, 0, 1 >$ are an orthonormal basis for $\mathbb{R}^3$, and $\mathbf{v}$ can be expressed as a linear combination of these vectors:

$$\mathbf{v} = v_1\hat{\mathbf{i}} + v_2\hat{\mathbf{j}} + v_3\hat{\mathbf{k}}$$

Similarly, for any orthonormal basis $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ we can write $\mathbf{v}$ as a linear combination

$$\mathbf{v} = (\mathbf{v} \cdot \mathbf{a}_1)\mathbf{a}_1 + (\mathbf{v} \cdot \mathbf{a}_2)\mathbf{a}_2 + (\mathbf{v} \cdot \mathbf{a}_3)\mathbf{a}_3.$$

Even more generally, we can do this for any orthogonal basis using our projection formulae. If $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ form a basis of orthogonal vectors, we can write $\mathbf{v}$ as a linear combination

$$\mathbf{v} = \left(\frac{\mathbf{v} \cdot \mathbf{a}_1}{\mathbf{a}_1 \cdot \mathbf{a}_1} \right) \mathbf{a}_1 + \left(\frac{\mathbf{v} \cdot \mathbf{a}_2}{\mathbf{a}_2 \cdot \mathbf{a}_2} \right) \mathbf{a}_2 + \left(\frac{\mathbf{v} \cdot \mathbf{a}_3}{\mathbf{a}_3 \cdot \mathbf{a}_3} \right) \mathbf{a}_3.$$

In this lab we're going to do something new. We're going to be rewriting vectors in terms of an orthonormal basis, but the vector space will consist of all 2π-periodic functions. You can verify this is a vector space. First, we need to find a basis. It turns out that this vector space is infinite dimensional.

An Orthogonal Basis

We first define an inner product of vectors in our vector space. If f and g are two 2π periodic function, we define their inner product to be:

$$< f, g >= \int_0^{2\pi} f(x)g(x)dx.$$

Now, I clain that a basis for the vector space is given by the following infinite list of functions:

$$\begin{aligned} &\cos{(mx)} \qquad \text{for } m = 0, 1, 2, \dots \\ &\sin{(nx)} \qquad \text{for } n = 1, 2, \dots \end{aligned}$$

These vectors are all 2π-periodic functions, so they are all in our vector space. As $n > 0$ they are all non-zero vectors.

These functions are linearly independent. In fact, they form an orthogonal set. It's a calculus II exercise to verify:

$$\begin{aligned} &\int_0^{2\pi} \cos{(mx)} \cos{(nx)} dx = 0 \qquad \text{for } m \neq n \\ &\int_0^{2\pi} \sin{(mx)} \sin{(nx)} dx = 0 \qquad \text{for } m \neq n \\ &\int_0^{2\pi} \cos{(mx)} \sin{(nx)} dx = 0 \end{aligned}$$

These vectors are not, however, of unit length. It's another calculus exercise to verify

$$\int_0^{2\pi} \cos(0x)\cos(0x)dx = \int_0^{2\pi} 1 \cdot 1 dx = 2\pi$$

$$\int_0^{2\pi} \cos(mx)\cos(mx)dx = \pi \quad \text{for } m \neq 0$$

$$\int_0^{2\pi} \sin(nx)\sin(nx)dx = \pi \quad \text{for } n \neq 0$$

That these vectors (functions) are a spanning set for a vector space of 2π-periodic functions is Fourier's Theorem. There are some technicalities we will ignore. It works.

Expressing a Function In Terms of the Basis

Given a "nice" 2π-periodic function $f(x)$, we want to express it in terms of the basis of sine and cosine functions. For each $m = 0, 1, \ldots$ and $n = 1, 2, \ldots$ we need to find the $\cos(mx)$ and $\sin(nx)$ parts of f. We do this with the usual projection formula.

For any $m = 0, 1, \ldots,$ the $\cos(mx)$ part of f is $a_m \cos(mx)$ where

$$a_m = \frac{< f, \cos(mx) >}{< \cos(mx), \cos(mx) >} = \begin{cases} \frac{1}{2\pi}\int_0^{2\pi} f\cos(mx)dx & m = 0 \\ \frac{1}{\pi}\int_0^{2\pi} f\cos(mx)dx & m > 0 \end{cases}$$

Similarly, for $n = 1, 2, \ldots$ the $\sin(nx)$ part of f is $b_n \sin(nx)$ where

$$b_n = \frac{< f, \sin(nx) >}{< \sin(nx), \sin(nx) >} = \frac{1}{\pi}\int_0^{2\pi} f\sin(nx)dx$$

Then the function f can be written as a sum of its parts as

$$f(x) \approx a_0\cos(0x) + a_1\cos(1x) + b_1\sin(1x) + a_2\cos(2x) + b_2\sin(2x) + \cdots$$

Here is an example. Let $f(x)$ be the 2π-periodic funcitons whose value is π between 0 and π and $-\pi$ from π to 2π. In Maple, define this piecewise so that it is correct on the interval from -2π to 2π and plot it:

```
f := x->piecewise(x <=-Pi,Pi,x<0,-Pi,x<=Pi,Pi,-Pi);

plot(f(x),x=-2Pi..2Pi,discont = false, thickness = 3);
```

The graph f is described as a “square wave”. When you see it, you’ll know why.

The sine parts we fine by computing

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin(nx)dx = \frac{4}{2n-1}$$

The fact that f is an odd function implies that all the cosine parts (the even parts) are 0.

So, we claim

$$f(x) \approx \frac{4}{1}\sin(1x) + \frac{4}{3}\sin(3x) + \frac{4}{5}\sin(5x) + \cdots$$

This is called the Fourier Series for our square wave, as as we take more and more terms, it gets closer and closer to approximating our function. Try plotting the first four Fourier Sums on the same axes with f:

plot([f(x),4sin(x)] x=-2Pi..2Pi,discont = false, color = [red,blue], thickness = [3,1]);
plot([f(x),4(sin(x) + 1/3 sin(3x))],x=-2Pi..2Pi,discont = false, color = [red,blue], thickness = [3,1];

etc...

Let's parse this Maple command: the first argument of the "plot" function is a list, in square brackets, of 2 functions to plot, separated by commas. The next argument says to display the plot on x-values from -2π to 2π (the y-values will be determined automatically). The remaining arguments specify options. "discont = false" says to draw the graphs as if they were continuous, even though f, for instance, is not continuous. "color" and "thickness" define these attributes for the graphs of the corresponding functions. The first function from the list will be graphed with color red and thickness 3, etc.

What You Should Hand In

This lab is just designed to be a fun "playing with Fourier series" lab. We won't do Fourier series in this class after this, but you'll definitely see them if you take 2280 next semester. It's cool to see them and how they work at some point.

What you should hand in is your Maple code, with your name commented at the top. Your Maple code should have a plot of the square wave, as well as plots of the first four Fourier sum approximations of the square wave.