

Math 2270 - Computer Lab 2 : Circuits

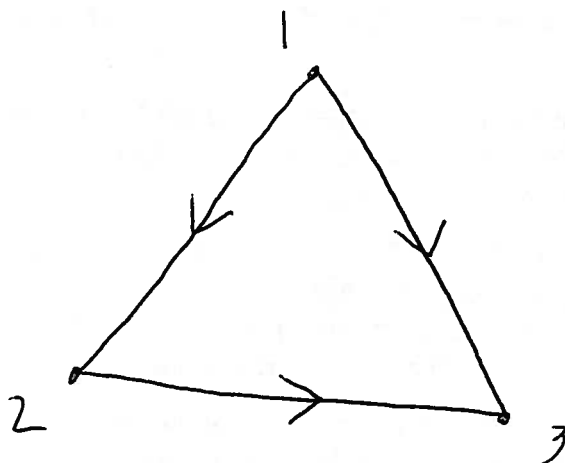
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In this lab we're going to investigate how linear algebra can be used and applied to solve questions involving circuit diagrams. All the background you need should be provided in this writeup, but more information can be found in section 8.2 of the textbook. We'll be returning to the ideas touched on in this lab, and exploring them in greater depth, towards the end of the semester.

1 Circuits and Matrices

Consider a graph consisting of vertices and directed edges. For example, the graph below of a triangle with vertices labeled 1, 2, 3 and edges from 1 to 2, from 1 to 3, and from 2 to 3.



To any directed graph we associated an incidence matrix A with one row for each edge, and one column for each vertex. If edge i goes from node j to node k then set the i, j -entry of A to -1 , the i, k -entry of A to 1 , and the other entries of row i to 0 .

The incidence matrix for our triangle will be:

$$A = \begin{pmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{pmatrix}$$

If we multiply A by a vector \mathbf{v} of dimension equal to (the number of columns of A = number of vertices of the graph), we get an output vector \mathbf{b} of dimension equal to (the number of rows of A = number of edges of the graph).

Imagine that the graph is an electrical circuit and the entries of the vector \mathbf{v} are the voltages at the corresponding vertices. Then in the example of our triangle we'd have:

$$A\mathbf{v} = \begin{pmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} v_2 - v_1 \\ v_3 - v_1 \\ v_3 - v_2 \end{pmatrix} = \mathbf{b}.$$

This vector \mathbf{b} gives us the voltage drop across each of our respective edges.

The column space of A is the collection of all linear combinations of the columns of A , which is the same as the collection of vectors $A\mathbf{v}$ of all possible voltage drops across our edges.

The nullspace of A will be the collection of all vertex voltages such that the voltage drop across every edge is 0 . This will only happen if all the voltages are the same. So, the nullspace is a one dimensional space consisting of all vectors \mathbf{v} with all entries the same.

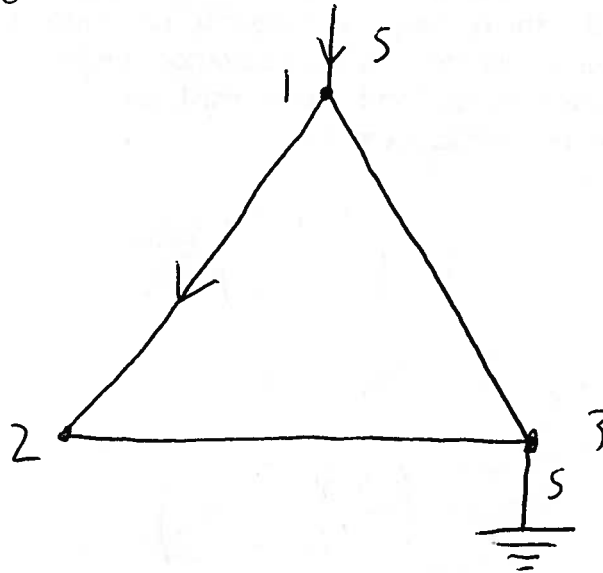
Now, Ohm's law says that the voltage drop across an edge is equal to the current through the edge multiplied by the resistance of the edge, or put more succinctly, $V = IR$. If we assume that each edge has resistance

1Ω (1 Ohm) then the current through the edge in Amps is equal to the voltage difference across the edge in Volts.

Kirchoff's Current Law says that the current flowing into a vertex equals the current flowing out, unless the vertex is a source or a sink. If the vector \mathbf{c} represents the current through each edge then the current flowing out of each vertex is $A^T \mathbf{c}$. For our triangle this is:

$$A^T \mathbf{c} = \begin{pmatrix} -1 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} -c_1 - c_2 \\ c_1 - c_3 \\ c_2 + c_3 \end{pmatrix}$$

Now, suppose that we introduce an external current s flowing into node 1 and a ground at node 3.



The grounded node has voltage zero, and has net current s flowing out. For the other vertices (only vertex 2 in the case of our triangle), the current flowing in matches the current flowing out, so our circuit satisfies:

$$A^T \mathbf{c} = \begin{pmatrix} s \\ 0 \\ -s \end{pmatrix}$$

The current vector \mathbf{c} , assuming the resistance across all edges is 1Ω , is $A\mathbf{v}$, so our equation above becomes:

$$A^T A \mathbf{v} = \begin{pmatrix} s \\ 0 \\ -s \end{pmatrix}$$

We can find the voltage at each node, given by the vector \mathbf{v} , by solving this equation. Now, we recall the nullspace of A is one dimensional, so there is exactly one free column in A , and one free variable in \mathbf{v} . Vertex 3 is grounded, so we set that free variable $v_3 = 0$, and we solve for a particular solution.

Finally, we could have different resistances across the edges. We can account for this by introducing a square, diagonal "resistance matrix" whose diagonal i, i entry is $\frac{1}{\Omega_i}$, where Ω_i is the resistance of edge i . For example, suppose in our triangle edges 1 and 2 have resistance 1Ω , while edge 3 has resistance 2Ω . Then our resistance matrix is

$$C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix}$$

To find the voltages we solve the equation:

$$A^T C A \begin{pmatrix} v_1 \\ v_2 \\ 0 \end{pmatrix} = \begin{pmatrix} s \\ 0 \\ -s \end{pmatrix}$$

Once the voltages are known we find the currents through each edge by computing $C A \mathbf{v}$.

For our particular problem if we set $s = 4$ the solution is:

$$\mathbf{v} = \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix}$$

Solving for v_1 and v_2 in Maple is pretty easy, but we must use Maple's "solve" command. We *cannot* solve the equation by taking the inverse of the matrix on the left, because the matrix on the left will always be singular.¹

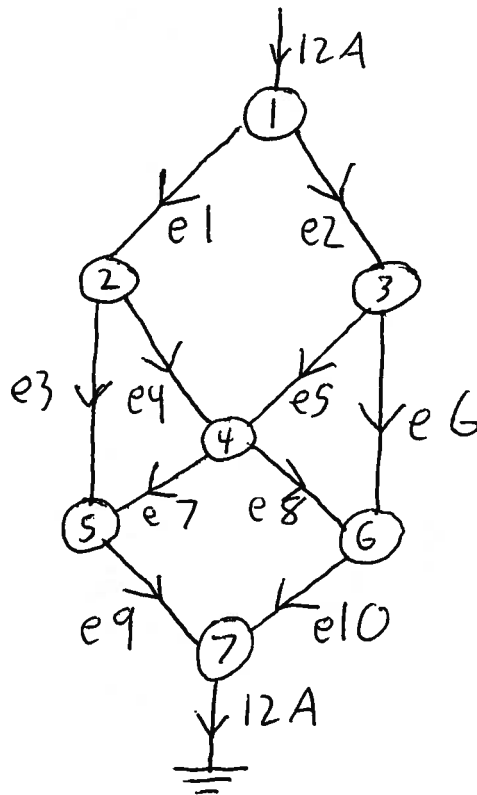
Maple code for solving for v_1 and v_2 , with $v_3 = 0$, is available on the class website. Please look through this code and read it for understanding, as it utilizes the same approach you'll apply for the more difficult problem below.

¹We know this because it has a non-trivial nullspace.

2 Maple Problem

For the following network diagram, find the voltage at each numbered node and the current flow across each numbered edge, given the following assumptions:

The resistance across each edge is 1Ω except for e_2 and e_9 , which have resistance 5Ω . There is a current of $12A$ that enters node 1 from the top, and a current of $12A$ that exits node 7 from the bottom. The voltage at node 7, v_7 , is set of 0.



For your writeup, you should hand in a printout with your name on it of the Maple code you use to answer the above question, along with, of course, the answer. The Maple code should look similar to the example code used to solve the triangle problem.