Math 2270 - Lecture 9: Inverse Matrices

Dylan Zwick

Fall 2012

This lecture covers **section 2.5** of the textbook.

1 The Idea of an Inverse

We'll start very simply. In arithmetic, we know $2 \times 5 = 10$. When 2 operates on 5 by multiplication, the output is 10. Now, suppose we know the output is 10, and we want to know the input. That is to say, we want to know what number, when we multiply it by 2, gives us 10. The way we do this is we divide 10 by 2 or, equivalently, multiply 10 by $\frac{1}{2}$. The operation of multiplying by $\frac{1}{2}$ is the inverse of the operation of multiplying by 2, and the product of 2 and $\frac{1}{2}$ is 1.

There was nothing particularly special about 2 here, and very similar reasoning with very similar results would apply for multiplication by 3, 20, π, or almost any real number we choose. I say *almost* because there is one very important exception. This exception is 0. If I tell you that the product of 0 and another number is 0, you can't tell me what that other number is! More precisely, that number could be *any* real number. On the other hand, if I tell you that the product of 0 and another number is 1, you'll have to call me a liar, because there is no number such that when it's multiplied by 0 the product is 1. So, the number 0 is not invertible!

A similar idea applies to matrices, but matrices are more complicated and more interesting. Suppose A is a square matrix, and $A\mathbf{x} = \mathbf{b}$, where $\mathbf{x}, \mathbf{b}$ are, as usual, vectors. We might be given $\mathbf{b}$ and asked to find $\mathbf{x}$. If there is another square matrix of the same size as A, call it A^{-1}, such that $A^{-1}A = I$ then we have $\mathbf{x} = A^{-1}A\mathbf{x} = A^{-1}\mathbf{b}$. So, if we know A^{-1} we can figure out $\mathbf{x}$ from $\mathbf{b}$. This is the analog in matrix multiplication for the situation described above in standard multiplication. Now, just as not all real numbers are invertible, with 0 being the exception, not all matrices are invertible. The question of *which* matrices are invertible, and how we can figure out if a given matrix is invertible, is one of the central questions in linear algebra.

Definition - A square matrix A is invertible if there exists a square matrix A^{-1} of the same size as A such that

$$A^{-1}A = I \qquad \text{and} \qquad AA^{-1} = I.$$

2 Some Properties of Inverse Matrices

We saw a few lectures ago that for a 2×2 matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

an inverse exsits if and only if $ad - bc \neq 0$. The number $ad - bc$ is called the *determinant* of this matrix, which is a concept about which we'll have much more to say later. The inverse of this matrix will be

$$A^{-1} = \frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

For example, the matrix

$$\begin{pmatrix} 3 & 5 \\ 1 & 2 \end{pmatrix}$$

has determinant $3 \times 2 - 5 \times 1 = 1$, is invertible, and has inverse

$$\begin{pmatrix} 2 & -5 \\ -1 & 3 \end{pmatrix}$$

If A and B are invertible matrices of the same size, then $A + B$ may or may not be invertible.

Example

1. Find invertible matrices A and B such that $A + B$ is not invertible.
2. Find singular matrices A and B such that $A + B$ is invertible.

1. $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$

$$A + B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

2. $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ $B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

$$A + B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

For products of matrices the situation is a little more straightforward. The product AB of two matrices A and B is invertible if and only if A and B are both themselves invertible. If this is the case, then the inverse of AB is

$$(AB)^{-1} = B^{-1}A^{-1}$$

Note that the order of the inverses (B^{-1} on the left, A^{-1} on the right) is the opposite of the order in the original product (A on the left, B on the right). This pattern generalizes. So

$$(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$$

and so on.

Finally, we note that if $BA = I$, then $B = A^{-1}$, and $AB = I$. So, if A has an inverse it is unique, and it is both the *left-inverse* and the *right-inverse*. This is important, and not super-quick to prove, and we won't have time to go over the proof in class. The proof is in the textbook, and I encourage you to read over it there.

3 Calculating Inverses Using Elimination

Suppose we're given the matrix

$$A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$$

and asked to find its inverse. We'll take it on faith for now that it is, in fact, invertible.[1] To calculate the inverse matrix we use the *Gauss-Jordan method*.

The Gauss-Jordan method takes our original matrix A and augments it with an identity matrix, producing in our example the 3×6 matrix

$$\begin{pmatrix} 2 & -1 & 0 & 1 & 0 & 0 \\ -1 & 2 & -1 & 0 & 1 & 0 \\ 0 & -1 & 2 & 0 & 0 & 1 \end{pmatrix}$$

We then perform elimination on the 3×3 matrix on the left, but every elimination step we apply to the entire 3×6 matrix. So, in our example, the first elimination step would be to add $\frac{1}{2}$ of row 1 to row 2 to get rid of the -1 term at the beginning of row 2. When we do this we get

[1]Finding the inverse is certainly one way to prove it's invertible!

$$\begin{pmatrix} 2 & -1 & 0 & 1 & 0 & 0 \\ 0 & \frac{3}{2} & -1 & \frac{1}{2} & 1 & 0 \\ 0 & -1 & 2 & 0 & 0 & 1 \end{pmatrix}$$

The next step in elimination is to use the $\frac{3}{2}$ pivot in the second row to eliminate the -1 term beneath it in the third row. To do this we add $\frac{2}{3}$ times the second row to the third, producing

$$\begin{pmatrix} 2 & -1 & 0 & 1 & 0 & 0 \\ 0 & \frac{3}{2} & -1 & \frac{1}{2} & 1 & 0 \\ 0 & 0 & \frac{4}{3} & \frac{1}{3} & \frac{2}{3} & 1 \end{pmatrix}$$

At this point standard elimination is over, but the Gauss-Jordan method is not done. The next thing we need to do is use our pivots to eliminate the non-zero terms *above* those pivots. This is something new. The first step in doing this would be to add $\frac{3}{4}$ row 3 to row 2, producing

$$\begin{pmatrix} 2 & -1 & 0 & 1 & 0 & 0 \\ 0 & \frac{3}{2} & 0 & \frac{3}{4} & \frac{3}{2} & \frac{3}{4} \\ 0 & 0 & \frac{4}{3} & \frac{1}{3} & \frac{2}{3} & 1 \end{pmatrix}$$

Finally, we add $\frac{2}{3}$ row 2 to row 1 to give us

$$\begin{pmatrix} 2 & 0 & 0 & \frac{3}{2} & 1 & \frac{1}{2} \\ 0 & \frac{3}{2} & 0 & \frac{3}{4} & \frac{3}{2} & \frac{3}{4} \\ 0 & 0 & \frac{4}{3} & \frac{1}{3} & \frac{2}{3} & 1 \end{pmatrix}$$

The 3×6 matrix above is now in something called "row echelon form". Our last step is we multiply each row by a constant so that its pivot becomes 1. In our example, this means multiplying row 1 by $\frac{1}{2}$, row 2 by $\frac{2}{3}$, and row 3 by $\frac{3}{4}$, producing

$$\begin{pmatrix} 1 & 0 & 0 & \frac{3}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & 1 & 0 & \frac{1}{2} & 1 & \frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{4} & \frac{1}{2} & \frac{3}{4} \end{pmatrix}$$

This matrix above is in something called "row reduced echelon form".[2] Now, let's see what we've done. We started with an augmented matrix with our matrix A on the left side and the identitity matrix I on the right, and we've transformed it into a matrix with the identity matrix I on the left, and what on the right? Well, the matrix on the right is A^{-1}! So,

$$A^{-1} = \begin{pmatrix} \frac{3}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & 1 & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} & \frac{3}{4} \end{pmatrix}$$

This procedure will work for more than just 3×3 matrices, and will in general work for $n \times n$ matrices. Note that an absolutely critical part of finding this determinant is that all the pivots had to be non-zero. If one of the pivots were zero (and there were no way we could switch rows to get around the problem) then the last step would fail. So, the inverse of a matrix exists if and only if elimination produced n non-zero pivots. The *determinant* of a matrix is the product of its pivots, and for a matrix to be invertible, it must have non-zero determinant.

[2]Or sometimes "reduced row echelon form". I've seen it both ways.

Example - Find the inverse of

$$B = \begin{pmatrix} 3 & 2 & 0 & 0 \\ 4 & 3 & 0 & 0 \\ 0 & 0 & 6 & 5 \\ 0 & 0 & 7 & 6 \end{pmatrix}$$

$$\left(\begin{array}{cccc|cccc} 3 & 2 & 0 & 0 & 1 & 0 & 0 & 0 \\ 4 & 3 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 6 & 5 & 0 & 0 & 1 & 0 \\ 0 & 0 & 7 & 6 & 0 & 0 & 0 & 1 \end{array}\right) \xrightarrow{\text{Subtract } \frac{4}{3} \text{ row 1 from row 2}} \left(\begin{array}{cccc|cccc} 3 & 2 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 & -\frac{4}{3} & 1 & 0 & 0 \\ 0 & 0 & 6 & 5 & 0 & 0 & 1 & 0 \\ 0 & 0 & 7 & 6 & 0 & 0 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{\text{Subtract } \frac{7}{6} \text{ row 3 from row 4}} \left(\begin{array}{cccc|cccc} 3 & 2 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 & -\frac{4}{3} & 1 & 0 & 0 \\ 0 & 0 & 6 & 5 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{6} & 0 & 0 & -\frac{7}{6} & 1 \end{array}\right) \xrightarrow{\text{Subtract } 6\times \text{ row 2 from row 1}} \left(\begin{array}{cccc|cccc} 3 & 0 & 0 & 0 & 9 & -6 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 & -\frac{4}{3} & 1 & 0 & 0 \\ 0 & 0 & 6 & 5 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{6} & 0 & 0 & -\frac{7}{6} & 1 \end{array}\right)$$

$$\xrightarrow{\text{Subtract } 30\times \text{ row 4 from row 3}} \left(\begin{array}{cccc|cccc} 3 & 0 & 0 & 0 & 9 & -6 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 & -\frac{4}{3} & 1 & 0 & 0 \\ 0 & 0 & 6 & 0 & 0 & 0 & 36 & -30 \\ 0 & 0 & 0 & \frac{1}{6} & 0 & 0 & -\frac{7}{6} & 1 \end{array}\right)$$

Multiply row 1 by $\frac{1}{3}$, row 2 by 3, row 3 by $\frac{1}{6}$, row 4 by 6

$$\longrightarrow \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 3 & -2 & 0 & 0 \\ 0 & 1 & 0 & 0 & -4 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 6 & -5 \\ 0 & 0 & 0 & 1 & 0 & 0 & -7 & 6 \end{array}\right)$$

$$\boxed{B^{-1} = \begin{pmatrix} 3 & -2 & 0 & 0 \\ -4 & 3 & 0 & 0 \\ 0 & 0 & 6 & -5 \\ 0 & 0 & -7 & 6 \end{pmatrix}}$$