Math 2270 - Lecture 8: Rules for Matrix Operations

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This lecture covers **section 2.4** of the textbook.

1 Matrix Basix

Most of this lecture is about formalizing rules and operations that we've already been using in the class up to this point. So, it should be mostly a review, but a necessary one. If any of this is new to you please make sure you understand it, as it is the foundation for everything else we'll be doing in this course!

A matrix is a rectangular array of numbers, and an "m by n" matrix, also written $m \times n$, has m rows and n columns. We can add two matrices if they are the same shape and size. Addition is termwise. We can also multiply any matrix A by a constant c, and this multiplication just multiplies every entry of A by c. For example:

$$\begin{pmatrix} 2 & 3 \\ 3 & 4 \\ 1 & 2 \end{pmatrix} + \begin{pmatrix} 3 & 5 \\ 1 & 0 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} 5 & 8 \\ 4 & 4 \\ 3 & 5 \end{pmatrix}$$

$$3\begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 2 & 4 \end{pmatrix} = \begin{pmatrix} 3 & 6 \\ 9 & 12 \\ 6 & 12 \end{pmatrix}$$

Moving on. Matrix multiplication is more tricky than matrix addition, because it isn't done termwise. In fact, if two matrices have the same size and shape, it's not necessarily true that you can multiply them. In fact, it's only true if that shape is *square*. In order to multiply two matrices A and B to get AB the number of *columns* of A must equal the number of rows of B. So, we could not, for example, multiply a 2×3 matrix by a 2×3 matrix. We could, however, multiply a 2×3 matrix by a 3×2 matrix.

If the number of columns in A is equal to the number of rows in B, then the product AB will be a matrix with the number of rows in A, and the number of columns in B. So, for example, a 2×3 matrix multiplied by a 3×2 matrix will produce a 2×2 matrix.

If the matrices A and B can be multiplied, then the entry in row i and column j of AB is the dot product of row i of A with column j of B. Expressed more mathematically

$$(AB)_{ij} = (\text{row } i \text{ of } A) \cdot (\text{column } j \text{ of } B).$$

So, as an example, for the matrices

$$A = \begin{pmatrix} 2 & 3 & 1 \\ 1 & 3 & 4 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \\ 1 & 2 & 1 \end{pmatrix}$$

The product BA does not make sense, but the product AB does, and is equal to

$$AB = \begin{pmatrix} 9 & 9 & 19 \\ 11 & 13 & 19 \end{pmatrix}$$

2 Laws of Matrix Arithmetic

Many of the standard rules from ordinary arithmetic carry over into matrix arithmetic. Some of these are[1]

$$
\begin{aligned}
A + B &= B + A \\
c(A + B) &= cA + cB \\
A + (B + C) &= (A + B) + C \\
C(A + B) &= CA + CB \\
(A + B)C &= AC + BC \\
A(BC) &= (AB)C
\end{aligned}
$$

Perhaps the most interesting, and unexpected, of the above rules is $A(BC) = (AB)C$. We call this *associativity*, and that matrix multiplication is associative isn't obvious from the definition of how matrices are multiplied, but it's true.

One rule from ordinary multiplication that is usually *not* true for matrix multiplication is

$$AB \neq BA$$

When you can switch the order of A and B in an equation like the one above, we say the operation is *commutative*. In general, matrix multiplication does **not** commute. For example

$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 2 & 2 \end{pmatrix} = \begin{pmatrix} 6 & 5 \\ 6 & 4 \end{pmatrix}$$

while

$$\begin{pmatrix} 2 & 1 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 4 & 5 \\ 6 & 5 \end{pmatrix} \neq \begin{pmatrix} 6 & 5 \\ 6 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 2 & 2 \end{pmatrix}$$

[1]We assume throughout that A, B, and C are matrices of a size and shape that the operations make sense.

Example - Show that $(A+B)^2$ is different from $A^2 + 2AB + B^2$, when

$$A = \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 0 \\ 3 & 0 \end{pmatrix}$$

Write down the correct rule for
$(A+B)(A+B) = A^2 + \underline{\ AB + BA\ } + B^2$.

$$A^2 = \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}$$

$$AB = \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 3 & 0 \end{pmatrix} = \begin{pmatrix} 7 & 0 \\ 0 & 0 \end{pmatrix}$$

$$BA = \begin{pmatrix} 1 & 0 \\ 3 & 0 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix}$$

$$B^2 = \begin{pmatrix} 1 & 0 \\ 3 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 3 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 3 & 0 \end{pmatrix}$$

$$A+B = \begin{pmatrix} 2 & 2 \\ 3 & 0 \end{pmatrix}$$

$$(A+B)^2 = \begin{pmatrix} 2 & 2 \\ 3 & 0 \end{pmatrix}\begin{pmatrix} 2 & 2 \\ 3 & 0 \end{pmatrix} = \begin{pmatrix} 10 & 4 \\ 6 & 6 \end{pmatrix}$$

$$A^2 + 2AB + B^2 = \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 14 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 3 & 0 \end{pmatrix} = \boxed{\begin{pmatrix} 16 & 2 \\ 3 & 0 \end{pmatrix} \neq \begin{pmatrix} 10 & 4 \\ 6 & 6 \end{pmatrix}}$$

$$A^2 + AB + BA + B^2 = \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 7 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 3 & 0 \end{pmatrix} = \begin{pmatrix} 10 & 4 \\ 6 & 6 \end{pmatrix}$$

3 Matrix Powers

We can take powers of matrices, but only if they're square. If A is a square matrix, then $A \cdot A$ is well-defined. If A is not square then $A \cdot A$ doesn't work for matrix multiplication. The usual rules for exponents, namely $A^p A^q = A^{p+q}$ and $(A^p)^q = A^{pq}$ still apply.

We define $A^0 = I$, where I is the identity matrix of the same size as A. We also define A^{-1} to be the inverse of A, so A^{-3} would be $A^{-1}A^{-1}A^{-1}$. Note that in usual arithmetic the inverse of a number exists *unless* the number is zero. For matrices whether the inverse exists or not is a tricky question, and the rules for figuring this out we will discuss in great detail later in this course.

Example - Compute A^2, A^3, A^4 for

$$A = \begin{pmatrix} 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$A^2 = \begin{pmatrix} 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$A^3 = \begin{pmatrix} 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 8 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$A^4 = \begin{pmatrix} 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 & 8 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

A matrix for which $A^n = 0$ for some n is called nilpotent.