

# Math 2270 - Lecture 7: Elimination Using Matrices

Dylan Zwick

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This lecture covers **section 2.3** of the textbook.

## 1 Elimination Matrices

Let's examine the following equation

$$\begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 4 \\ 7 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 7 \end{pmatrix}$$

The matrix on the left is acting on the vector  $\begin{pmatrix} 1 \\ 4 \\ 7 \end{pmatrix}$  by subtracting from the second entry 2 times the first entry. If we have a different vector on the left we get a different equation

$$\begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 8 \\ 6 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix}$$

but the same action. Again, the action of the matrix is that it subtracts 2 times the first entry from the second entry. We'll call the matrix on the

left an *elimination matrix*, a term which we'll define precisely a bit later in this lecture.

Now, let's take a look at the system of three equations with three unknowns we studied at the end of the last section

$$\begin{aligned}2x + 4y - 2z &= 2 \\4x + 9y - 3z &= 8 \\-2x - 3y + 7z &= 10\end{aligned}$$

We can rewrite this as a matrix equation

$$\begin{pmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 8 \\ 10 \end{pmatrix}$$

If we left-multiply the coefficient matrix on the left by our elimination matrix we get

$$\begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{pmatrix} = \begin{pmatrix} 2 & 4 & -2 \\ 0 & 1 & 1 \\ -2 & -3 & 7 \end{pmatrix}.$$

The action of the elimination matrix on the matrix of coefficients is it subtracts from the second row 2 times the first row. It's essentially the same action that it had on the column vectors.

Let's go back to the matrix form of our system of linear equations.

$$\begin{pmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 8 \\ 10 \end{pmatrix}$$

If we left-multiply both sides of this equation by our elimination matrix we get

$$\begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 8 \\ 10 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 2 & 4 & -2 \\ 0 & 1 & 1 \\ -2 & -3 & 7 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ 10 \end{pmatrix}$$

The above equation is the matrix form of the system of linear equations

$$\begin{aligned} 2x + 4y - 2z &= 2 \\ 0x + 1y + 1z &= 4 \\ -2x - 3y + 7z &= 10 \end{aligned}$$

The above system of linear equations is the system we get if we start with our original system, and subtract two times the first equation from the second. This is the first step in Gaussian elimination! What this means is that multiplication by an elimination matrix is the matrix version of an elimination step in Gaussian elimination.

We should probably formally define an elimination matrix before we get any farther.

**Definition** - An *elimination matrix* is a matrix that subtracts a multiple  $\ell$  of row  $j$  from row  $i$ . It is the same as the identity matrix except for a nonzero entry,  $-\ell$ , in the  $i, j$  position.

Example - Find a matrix  $L$  that converts the matrix form of the system of linear equations

$$\begin{aligned} 2x + 4y - 2z &= 2 \\ 4x + 9y - 3z &= 8 \\ -2x - 3y + 7z &= 10 \end{aligned}$$

into an upper triangular system.

$$\begin{pmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 8 \\ 10 \end{pmatrix}$$

Subtract 2x first row from second:

$$\begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 8 \\ 10 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 4 & -2 \\ 0 & 1 & 1 \\ -2 & -3 & 7 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ 10 \end{pmatrix}$$

Add first row to third row:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 4 & -2 \\ 0 & 1 & 1 \\ -2 & -3 & 7 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 4 \\ 10 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 4 & -2 \\ 0 & 1 & 1 \\ 0 & 1 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ 12 \end{pmatrix}$$

Subtract row 2 from row 3.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 4 & -2 \\ 0 & 1 & 1 \\ 0 & 1 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 4 \\ 12 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 4 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ 8 \end{pmatrix} \Rightarrow$$

$$\begin{aligned} 2x + 4y - 2z &= 2 \\ y + z &= 4 \\ 4z &= 8 \end{aligned}$$

Upper Triangular

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \boxed{\begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 3 & -1 & 1 \end{pmatrix}}$$

## 2 Permutation Matrices

If we recall our elimination steps, it's possible that we have to switch or "permute" two rows. We can do this with matrices as well. For example, examine the following equation

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 5 \end{pmatrix}$$

The action of the matrix on the left is that it permutes the first and second entries of the vector. It has the same effect on a matrix

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{pmatrix} = \begin{pmatrix} 4 & 9 & -3 \\ 2 & 4 & -2 \\ -2 & -3 & 7 \end{pmatrix}$$

So, Gaussian elimination can be performed by a series of multiplications by elimination matrices and permutation matrices. We should probably formally define a permutation matrix.

**Definition** - The permutation matrix  $P_{ij}$  is the identity matrix with rows  $i$  and  $j$  reversed. When it left-multiplies another matrix, it exchanges rows  $i$  and  $j$ .

Example - Convert the following system into of linear equations into matrix form, and perform Gaussian elimination by a series of matrix multiplications.

$$\begin{aligned} 0x + 2y &= 4 \\ 3x - 2y &= 5 \end{aligned}$$

$$\begin{pmatrix} 0 & 2 \\ 3 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 2 \\ 3 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 4 \\ 5 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 3 & -2 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 \\ 4 \end{pmatrix}$$

↑  
Upper triangular.

$$3x - 2y = 5$$

$$2y = 4$$

$$\Rightarrow \boxed{y = 2, \quad x = 3}$$