

Math 2270 - Lecture 6: Introduction to Elimination

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This lecture covers **section 2.2** of the textbook.

1 Elimination for Equations Involving Two Variables

We're going to start our discussion of elimination, also known as Gaussian elimination after famous smart guy Carl Gauss, by looking at a situation we've already encountered in this class, and you've probably encountered a number of times in previous math classes. However, we're going to look at it in a slightly different way, and then see how this way generalizes to larger situations.

What we're going to look at is solving a system of two equations with two unknowns. We'll use as our running example the system

$$\begin{aligned}x - 2y &= 1 \\3x + 2y &= 11\end{aligned}$$

We want to find values of x and y that satisfy both of the above equations simultaneously. Now, a basic rule of algebra is that you can add the same thing to both sides of an equation. So, for example, if two sides of an equation are equal, they're still equal if you add 2 to both sides. Focusing on our system above, if the values of x and y are such that $x - 2y = 1$, then

adding $x - 2y$ to one side of an equation and adding 1 to another side of the equation maintains equality.

For our two equations above, we can multiply the first equation by -3 on both sides, and add this multiple to the second equation to get

$$8y = 8$$

We have transformed our second equation, originally $3x + 2y = 11$, into a new equation, $8y = 8$, that must also be true. From this new equation we see right away that $y = 1$. If we substitute $y = 1$ back into our first equation, $x - 2y = 1$, we get $x = 3$.

After we've transformed our second equation, we can write the transformed second equation, along with our first equation, as

$$\begin{array}{rcl} x - 2y & = & 1 \\ 8y & = & 8 \end{array}$$

For obvious reasons, a system of equations that looks like the one above is called an *upper triangular system*. The goal of elimination is to produce an upper triangular system. An upper triangular system can easily be solved from the bottom upwards using a process called *back substitution*.

We call the coefficient of x in the first equation the *first pivot*. We use this x coefficient to eliminate the x term in the second equation, and we call the coefficient of y in the second equation the *second pivot*. In general a pivot is the first nonzero in a row that does the elimination. The pivots are the coefficients that appear on the diagonal of the triangle in the upper triangular system we get after performing elimination.

Example - Convert the system below into an upper triangular system, and use back substitution to solve for x and y :

$$\begin{aligned} 2x - 5y &= 19 \\ -4x + 7y &= -29 \end{aligned}$$

$2x$ first row to second

$$\Rightarrow \begin{cases} 2x - 5y = 19 \\ -3y = 9 \end{cases}$$

$$\Rightarrow y = -3 \quad 2x - 5(-3) = 19$$

$$2x = 4$$

$$x = 2$$

$$\begin{cases} y = -3 \\ x = 2 \end{cases}$$

2 When Elimination Doesn't Work

Elimination doesn't always work. In particular, you can have situations where you run out of pivots. If this happens, there are either no solutions, or infinitely many solutions, but there will not be one and only one solution.

If we perform elimination on the following system of equations

$$\begin{aligned} x - 2y &= 1 \\ 3x - 6y &= 11 \end{aligned}$$

we subtract 3 times the first equation from the second to get

$$\begin{aligned} x - 2y &= 1 \\ 0y &= 8 \end{aligned}$$

There is *no* value of y for which the second equation is true, and therefore there is no solution.

On the other hand, suppose we have the following system of equations

$$\begin{aligned}x - 2y &= 1 \\ 3x - 6y &= 3\end{aligned}$$

If we perform elimination here we again subtract three times the first row from the second, but this time when we do this we get

$$\begin{aligned}x - 2y &= 1 \\ 0y &= 0\end{aligned}$$

The second equation in the system above is *always* true. So, we can pick y arbitrarily, and then solve for x in the first equation. As y can be any real number, and there are an infinite number of real numbers, there are an infinite number of solutions.

Finally, we can have situations where elimination works, but we need to switch up the equations. For example, if we have the system

$$\begin{aligned}0x + 2y &= 4 \\ 3x - 2y &= 5\end{aligned}$$

the first pivot position contains a 0. This is not allowed, as 0 cannot be a pivot. But, this is fixable. We just switch the order of the rows to get the system

$$\begin{aligned}3x - 2y &= 5 \\ 2y &= 4\end{aligned}$$

which is not only solvable, but already upper triangular!

Example - For which numbers a does elimination break down (1) permanently (2) temporarily?

$$\begin{aligned} ax + 3y &= -3 \\ 4x + 6y &= 6 \end{aligned}$$

If $a = 2$ then we get

$$\begin{aligned} 2x + 3y &= -3 \\ 4x + 6y &= 6 \end{aligned} \Rightarrow \begin{aligned} 2x + 3y &= -3 \\ 0y &= 12 \end{aligned}$$

No solutions

Breaks down permanently if $a = 2$.

If $a = 0$

$$\begin{aligned} 0x + 3y &= -3 \\ 4x + 6y &= 6 \end{aligned} \xrightarrow{\substack{\text{Switch} \\ \text{Rows}}} \begin{aligned} 4x + 6y &= 6 \\ 3y &= -3 \end{aligned}$$

Breaks down temporarily if $a = 0$. In upper triangular form.

3 Elimination with Three Equations and Three Unknowns

The methodology we used for the two equations with two unknowns situation works for three equations with three unknowns, or in general for n equations with n unknowns. The basic iterative process is:

1. Find a pivot without all 0 terms beneath it. Exchange rows if necessary.
2. Eliminate all the terms beneath the pivot.
3. If the matrix isn't upper triangular, repeat steps 1 and 2 until it is.

Let's run through this process for the following system of three equations with three unknowns:

$$\begin{aligned}2x + 4y - 2z &= 2 \\4x + 9y - 3z &= 8 \\-2x - 3y + 7z &= 10\end{aligned}$$

Our first pivot is the $2x$ term in the first row. We can eliminate the x terms in the second and third rows by subtracting 2 times the first equation from the second, and adding the first equation to the third. If we do this we get the system:

$$\begin{aligned}2x + 4y - 2z &= 2 \\1y + 1z &= 4 \\1y + 5z &= 12\end{aligned}$$

If we then take this new system and subtract the second line from the third we get:

$$\begin{aligned}2x + 4y - 2z &= 2 \\1y + 1z &= 4 \\4z &= 8\end{aligned}$$

This is an upper triangular system, and we can use back substitution to get $z = 2, y = 2, x = -1$.

Example - Apply elimination (circle the pivots) and use back substitution to solve

$$\begin{array}{rcl} 2x - 3y & & = 3 \\ 4x - 5y + z & = & 7 \\ 2x - y - 3z & = & 5 \end{array}$$

Subtract $2x$ row 1 from row 2.

$$\begin{array}{rcl} 2x - 3y & & = 3 \\ & y + z & = 1 \end{array}$$

$$2x - y - 3z = 5$$

Subtract row 1 from row 3.

$$\begin{array}{rcl} 2x - 3y & & = 3 \\ & y + z & = 1 \\ & 2y - 3z & = 2 \end{array}$$

Subtract $2x$ row 2 from row 3.

$$\begin{array}{rcl} 2x - 3y & & = 3 \\ & y + z & = 1 \\ & -5z & = 0 \end{array}$$

$$z = 0$$

$$y + 0 = 1 \Rightarrow y = 1$$

$$2x - 3(1) = 3 \Rightarrow x = 3$$

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$$\boxed{x=3, y=1, z=0}$$

