# Math 2270 - Lecture 5: Vectors and Linear Equations in Three Dimensions

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Fall 2012

1 Intersecting Planes

We mentioned at the end of the last lecture that we can represent a plane in $\mathbb{R}^3$ as the set of all points (x, y, z) that satisfy an equation of the form $ax + by + cz = d$ where a, b, c are not all 0. Perhaps the most straightforward planes are the coordinate planes, defined by $x = 0$, $y = 0$, and $z = 0$, respectively. These are drawn below:
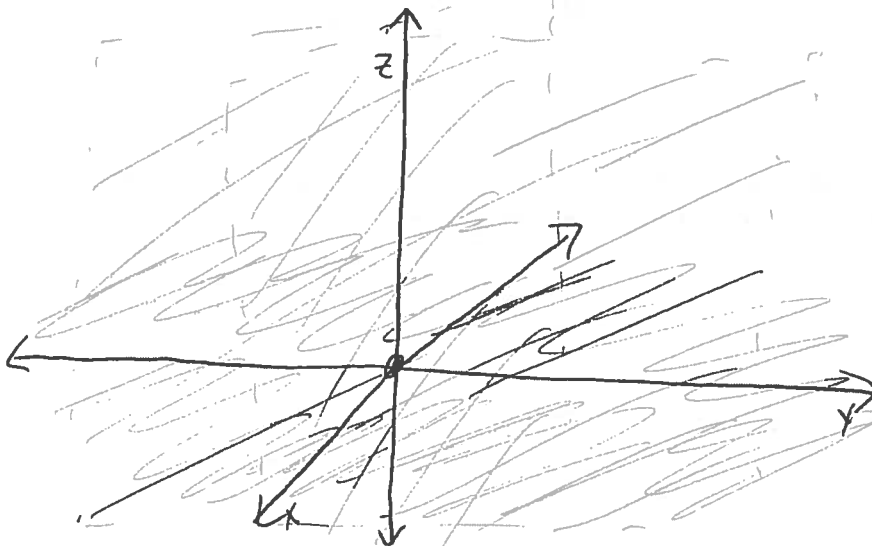
Drawing 3 planes is hard in 2 dimensions.

Now, the coordinate planes intersect at exactly one point, $x = y = z = 0$, a.k.a. the origin. I've chosen to examine the coordinate planes for

one very good reason, *they're easy to draw*! But, the idea that three planes intersect at a point does hold true for most planes.

Let's return to the system of equations we examined at the end of the last lecture.

$$\begin{aligned} x &+ 2y &+ 3z &= 6 \\ 2x &+ 5y &+ 2z &= 4 \\ 6x &- 3y &+ z &= 2 \end{aligned}.$$

Each of these equations represents a plane in three-dimensional space. The first two planes intersect each other in a line **L**.

We can represent this line parametrically by first finding two points on the line. Setting $z = 0$ we can solve for x and y in the first two equations to get $x = 22, y = -8$, so the two planes intersect at the point $(22, -8, 0)$. Setting $y = 0$ we can solve for x and z in the first two equations to get $x = 0, z = 2$, so the two planes intersect at the point $(0, 0, 2)$. The two planes will in fact intersect along the entire line through these points, which we can write parametrically as:

$$\begin{aligned} x(t) &= 22t \\ y(t) &= -8t \\ z(t) &= 2 - 2t \end{aligned}.$$

So, for example, if we set $t = -1$ we get the point $(-22, 8, 4)$, which will also be a point on both of the first two planes.

Example - Find parametric equations for the line of intersection of the second and third plane.

$z = 0$

$2x + 5y = 4$

$6x - 3y = 2$

$\Rightarrow -18y = -10$

$y = \frac{5}{9}$

$6x - 3\left(\frac{5}{9}\right) = 2$

$6x = \frac{6}{3} + \frac{5}{3} = \frac{11}{3}$

$x = \frac{11}{18}$

$P_1 = \left(\frac{11}{18}, \frac{5}{9}, 0\right)$

$x = 0$

$5y + 2z = 4$

$-3y + z = 2$

$11y = 0$

$y = 0$

$z = 2$

$P_2 = (0, 0, 2)$

$$\boxed{\begin{array}{l} X(t) = \frac{11}{18} t \\[2mm] y(t) = \frac{5}{9} t \\[2mm] z(t) = 2 - 2t \end{array}}$$

Example - Find a point, if one exists, where these two lines of intersection themselves intersect.

Want s, t such that:

$\frac{11}{18} s = 22t$

$\frac{5}{9} s = -8t$

$2 - 2s = 2 - 2t \Rightarrow s = t$

$\frac{5}{9} t = -8t \Rightarrow t = 0,$

$\Rightarrow$ At $s = t = 0$ intersect at point $(0, 0, 2)$, which is where our 3 planes intersect.

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Note that it is not always the case that two planes intersect in a line. If the two planes are in fact the same, then their intersection is, obviously, an entire plane. If the two planes are parallel, then they won't intersect at all. For example, the planes defined by

$$
\begin{aligned}
x + 2y + 3z &= 6 \\
x + 2y + 3z &= 0
\end{aligned}
$$

do not intersect anywhere.[1] This is because the two planes are parallel.

2 The Column Point of View

Just as we did with two lines in a plane, we can rewrite our three plane equations in terms of a linear combination of three columns:

$$
x \begin{pmatrix} 1 \\ 2 \\ 6 \end{pmatrix} + y \begin{pmatrix} 2 \\ 5 \\ -3 \end{pmatrix} + z \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 6 \\ 4 \\ 2 \end{pmatrix}.
$$

From this point of view, we're trying to find the coefficients x, y, z such that the above equality is true. When viewed from this perspective, the answer $(0, 0, 2)$ practically jumps out at you!

When can represent this situation geometrically as:
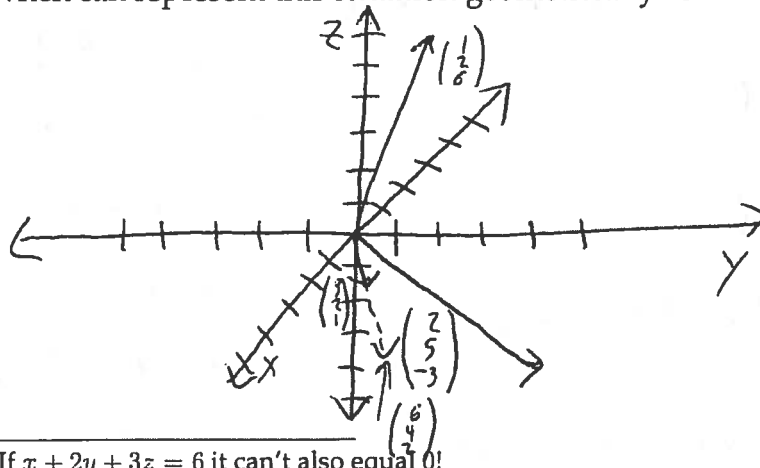
[1] If $x + 2y + 3z = 6$ it can't also equal 0!

We note that linear combinations of the above three columns span all of $\mathbb{R}^3$.

3 The Matrix Point of View

Finally, we can rewrite our three linear equations, or our three vector equations[2], as a matrix equation:

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 2 \\ 6 & -3 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ 4 \\ 2 \end{pmatrix}$$

and the problem of figuring out the point (x, y, z) where our three planes intersect is the problem of figuring out the input vector $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ that gives us the output $\begin{pmatrix} 6 \\ 4 \\ 2 \end{pmatrix}$.

Example - Find the input vector $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ that produces the given output vector:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix}.$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix}$$

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[2]Fundamentally the same thing.

The matrix on the left is a very special matrix called the *identity matrix*, or the 3×3 identity matrix if you need to specify the dimension. It's the one and only matrix for which the ouput always equals the input. In this way it's analagous to multiplication by 1 or addition by 0 in standard arithmetic.