

Math 2270 - Lecture 4: Vectors and Linear Equations

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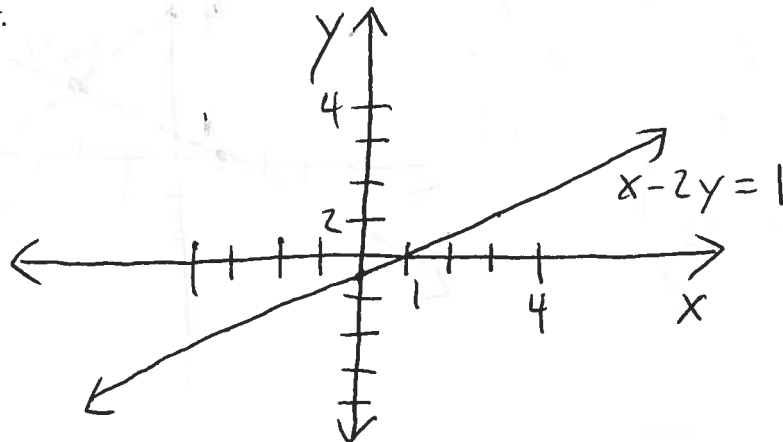
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1 Intersecting Lines

A fundamental idea in plane geometry, going all the way back to Euclid, is that two lines in a plane, unless they're parallel, intersect at one point. Understanding this idea algebraically is the beginning of linear algebra.

Mathematically, we can define a line in \mathbb{R}^2 with an equation of the form $ax + by = c$, where the coefficients a and b are not both zero. The line is going to be the set of all values (x, y) such that the equality is satisfied.¹

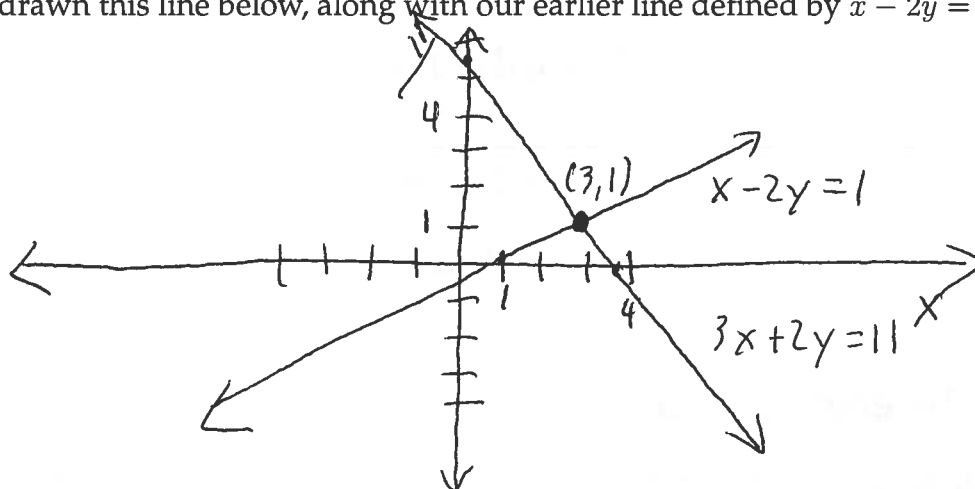
For example, the equation $x - 2y = 1$ defines a line, which we've drawn below.



¹By "satisfy the equality" we mean the set of x and y values that make the equality true.

Every point on the line² satisfies the equality, while every point not on the line³ does not.⁴

Now, let's take a look at another line, the line $3x + 2y = 11$. We've drawn this line below, along with our earlier line defined by $x - 2y = 1$.

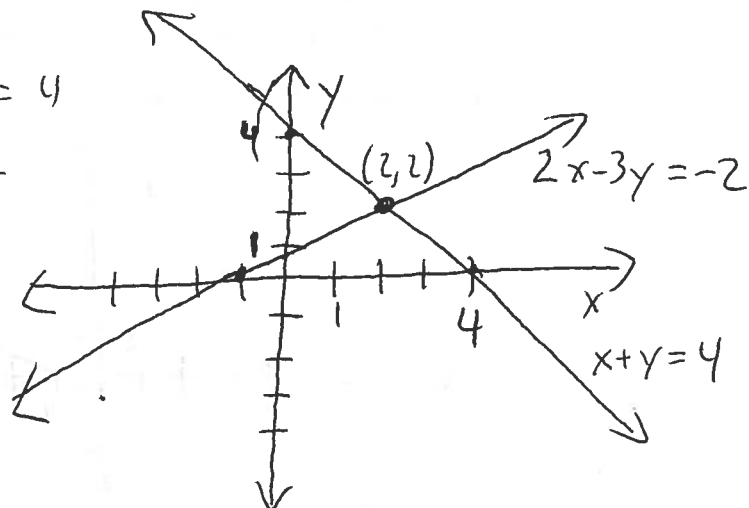


The two lines meet at a point, $(3, 1)$, and this point corresponds with the only values of x and y ($x = 3, y = 1$) that satisfy both equations.

Example - At what point do the lines $x + y = 4$ and $2x - 3y = -2$ intersect?

$$\begin{aligned} x + y &= 4 \\ \Rightarrow y &= 4 - x \\ 2x - 3y &= -2 \\ 2x - 3(4 - x) &= -2 \\ 5x &= 10 \\ x &= 2 \end{aligned}$$

$$\begin{aligned} 2 + y &= 4 \\ y &= 2 \\ (2, 2) \end{aligned}$$



²For example the point $(3, 1)$.
³For example the point $(0, 0)$.
⁴A fancy mathematical way of saying this is that the line is the *locus* of points satisfying the equation.

Example - At what point do the lines $x + y = 4$ and $x + y = 12$ intersect?

$$y = 4 - x$$

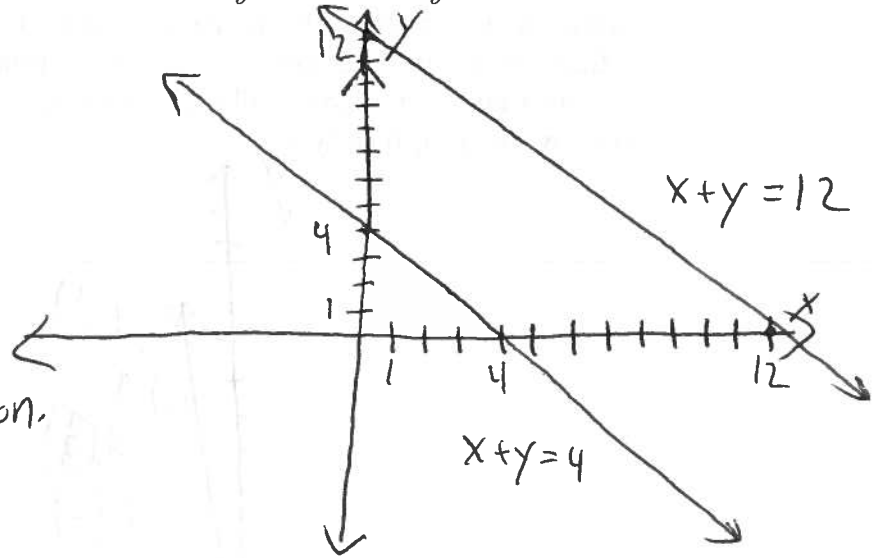
$$x + (4 - x) = 12$$

$$4 = 12?$$

Not true!

So, no solution.

(The lines are parallel.)



Now, what does this have to do with vectors? Well, we note that we can combine the two equations that define our lines using vectors, and write them as a linear combination.

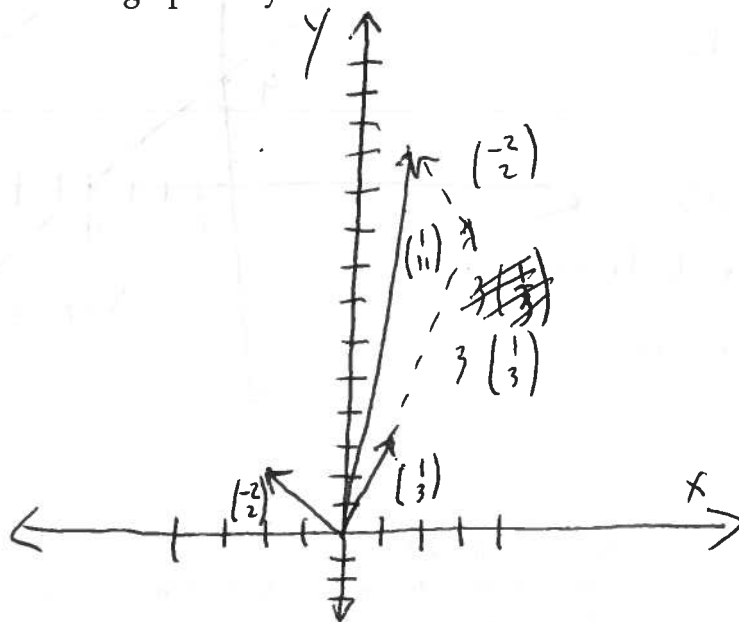
We can write:

$$\begin{aligned}x - 2y &= 1 \\3x + 2y &= 11\end{aligned}$$

as

$$x \begin{pmatrix} 1 \\ 3 \end{pmatrix} + y \begin{pmatrix} -2 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 11 \end{pmatrix}.$$

We're looking for the values of x and y that satisfy the above equation. Another way of putting this is, we're looking for the linear combination of the first two column vectors that equals the third. In this case, there's only one, and that's the linear combination we get when $x = 3$ and $y = 1$. We can view this graphically as:



2 When Two Lines Intersect At a Unique Point

Now, suppose we have two lines $ax + by = p$, and $cx + dy = q$. Here a, b, c, d, p, q are real numbers. We know from geometry that the two lines intersect at a unique point if and only if the two lines aren't parallel. If the two lines are parallel, that means they have the same slope.⁵ The slope of the first line is $-a/b$, while the slope of the second is $-c/d$. If the two slopes are equal then:

$$\begin{aligned} -a/b &= -c/d \\ \Rightarrow ad &= bc \\ \Rightarrow ad - bc &= 0. \end{aligned}$$

⁵We won't be fastidious here and worry about whether $b = 0$, although we probably should be. Just know that our final statement about the determinant is true even if $b = 0$, but this requires a little more bookkeeping.

So, the two lines intersect at a unique point if and only if $ad - bc = 0$.

Going back to our vector equation, we can view the problem of figuring out the point of intersection as the problem of figuring out the x and y values such that

$$x \begin{pmatrix} a \\ c \end{pmatrix} + y \begin{pmatrix} b \\ d \end{pmatrix} = \begin{pmatrix} p \\ q \end{pmatrix}.$$

We can rewrite this yet again in terms of matrix multiplication as:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} p \\ q \end{pmatrix}.$$

The problem of finding our point of intersection becomes the problem of finding the input vector $\begin{pmatrix} x \\ y \end{pmatrix}$ that gives us the output vector $\begin{pmatrix} p \\ q \end{pmatrix}$. In lecture 3 we learned this was only possible if the coefficient matrix is invertible. We just learned this is also only possible if $ad - bc \neq 0$. So, the coefficient matrix is invertible if and only if $ad - bc \neq 0$. We call the number $ad - bc$ the *determinant* of the coefficient matrix. Much more on this later.

3 Intersecting Planes

Two planes will, unless they're parallel, intersect along a line. A plane and a line will, unless they're parallel, intersect at a point. So, three planes will, usually⁶ intersect at a point.

A plane in \mathbb{R}^3 is represented by an equation of the form $ax + by + cz = d$, where a, b, c, d are constants and it's not the case that $a = b = c = 0$. The plane will be all the values of x, y, z that satisfy the equation.

⁶We can make the term "usually" mathematically precise, by saying outside of a set of measure 0, but I think it's clear what I mean here by "usually".

Example - Calculate the values of x, y, z that satisfy the three linear equations below:

$$\begin{aligned}x + 2y + 3z &= 6 \\2x + 5y + 2z &= 4 \\6x - 3y + z &= 2\end{aligned}$$

$$\begin{aligned}x &= 6 - 2y - 3z \\2(6 - 2y - 3z) + 5y + 2z &= 4 \\12 + y - 4z &= 4 \\y &= 4z - 8 \\x &= 6 - 2(4z - 8) - 3z \\&= 6 - 8z + 16 - 3z \\&= 22 - 11z \\6(22 - 11z) - 3(4z - 8) + z &= 2 \\132 - 66z - 12z + 24 + z &= 2 \\156 - 77z &= 2\end{aligned}$$

$$\begin{aligned}154 &= 77z \\ \Rightarrow z &= 2 \\ y &= 4(2) - 8 = 0 \\ x &= 6 - 2(0) - 3(2) = 0.\end{aligned}$$

$x = 0$
$y = 0$
$z = 2$

We just calculated the point where the three planes defined by the three equations above intersect. We'll talk more about intersecting planes in lecture 5.