Math 2270 - Lecture 3: Matrices

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1 Matrix Multiplication

Suppose we have three vectors in $\mathbb{R}^3$, given below:

$$\mathbf{u} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \mathbf{v} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \mathbf{w} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

The linear combinations of these three vectors can be written as

$$c\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + d\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} + e\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} c \\ d - c \\ e - d \end{pmatrix}.$$

Example - What is the above linear combination with

$$c = 1, d = 4, \text{ and } e = 9?$$

$$1\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + 4\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} + 9\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix}$$

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Now, we note that using the standard rules of matrix multiplication (which I assume you all know and have seen before) we can write this above product as:

$$\begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} c \\ d \\ e \end{pmatrix} = \begin{pmatrix} c \\ d - c \\ e - d \end{pmatrix}.$$

Call the matrix on the left A. What we've done here is a bit sneaky. We've changed our point of view. Instead of three scalars acting on a linear combination of vectors, we have a matrix acting upon the vector of those scalars! (Reread that last sentence if you must, but it does make sense.)

If we rewrite c, d, e as x_1, x_2, x_3 we get:

$$\begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 - x_1 \\ x_3 - x_2 \end{pmatrix}.$$

We see that our output vector is just the successive differences of the input vector. We call the matrix causing this action, the matrix A, a "difference matrix".

2   Linear Equations and a Matrix Inverse

Suppose instead of knowing the inputs x_1, x_2, x_3 above, all we know are the outputs. So, we know that after our matrix acts on the vector $\begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix}$ the output is $\begin{pmatrix} b_1 & b_2 & b_3 \end{pmatrix}$. Given the output, can we figure out the input?

It turns out in the particular situation of our example the answer is yes, but this isn't always the case. In our example we have:

$$x_1 = b_1$$
$$x_2 - x_1 = b_2$$
$$x_3 - x_2 = b_3.$$

Our difference matrix is *lower triangular*, which is a concept we'll be seeing again. The important fact about a lower triangular matrix for our purposes right now is that solving our inverse problem (for a given output, what is the input) is very straightforward. We can see just by a couple simple substitutions that:

$$x_1 = b_1$$
$$x_2 = b_1 + b_2$$
$$x_3 = b_1 + b_2 + b_3.$$

Example - If the output vector from our difference matrix is $\begin{pmatrix} 1 & 1 & 1 \end{pmatrix}$ what was the input?

$$X_1 = 1$$

$$X_2 = 1 + 1 = 2$$

$$X_3 = 1 + 1 + 1 = 3$$

$$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

We can rewrite our system of equations above in matrix form:

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

Call the matrix on the left S. So, what we have is that our matrix A takes an input vector $\mathbf{x}$ and gives us an output $\mathbf{b}$, while the matrix S takes the output vector $\mathbf{b}$ and gives us back the input vector $\mathbf{x}$. In other words, the matrix S *inverts* the action of A on $\mathbf{x}$. We call S the *inverse* of A, and we write this $S = A^{-1}$.

Questions about how, in general, we find this inverse matrix will be a major theme of this course. However, first we must address the question of whether the inverse matrix always exists. The answer is no.

Take as our example the matrix equation (we'll call the matrix on the left C):

$$\begin{pmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 - x_3 \\ x_2 - x_1 \\ x_3 - x_2 \end{pmatrix}.$$

Example - What is the output when the input vector is $(\ 0 \ \ 0 \ \ 0\ )$? What about when the input vector is $(\ 2 \ \ 2 \ \ 2\ )$?

$$\begin{pmatrix} 0-0 \\ 0-0 \\ 0-0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} 2-2 \\ 2-2 \\ 2-2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\underline{Both} \quad \boxed{\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}}$$

As the above example illustrates, it's possible for there to be more than one input with the same output. This means that an inverse matrix can't exist. If both x_1 and x_2 have output b, then what is the input that generates b? There's no single answer to that question.

3   Independence and Dependence

What is going on with A and C is that the colums of A are *independent*. This means that there is no way of writing one column of A as a linear combination of the other two. In this situation the columns of A span a three-dimensional space, and any point in that three dimensional space can be written *uniquely* as a linear combination of the columns of A.

For the matrix C, on the other hand, any column can be written as a linear combination of the other two. This means the columns of C span, at most, a two-dimensional plane in three dimensional space. So, for any point not on this plane, there is no way a linear combination of the three

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columns of C can equal it, while for any point on this plane, there are an infinite number of ways. We call a matrix that is not invertible *singular*.

Wouldn't it be nice if there were a way of telling if a given matrix A were invertible? Well, stick around. There is, and we'll be delving much deeper into that question later in the course.

Example - Find a way of writing the first column of C as a linear combination of the other two.

$$\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = -1 \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} - 1 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$