Math 2270 - Lecture 39 : Graphs and Incidence Matrices

Dylan Zwick

Fall 2012

This lecture covers section *section 8.2* of the textbook.

Today we're going to review some facts about graph theory and their associated incidence matrices. We've seen this before, in computer lab 2, but today we're going to talk about it in a little more depth, particularly now that we know about the four fundamental subspaces.

The assigned problems for this section are

Section 8.2 - 1, 2, 3, 4, 5

Graphs and Incidence Matrices

A graph is a finite collection of nodes connected by edges.
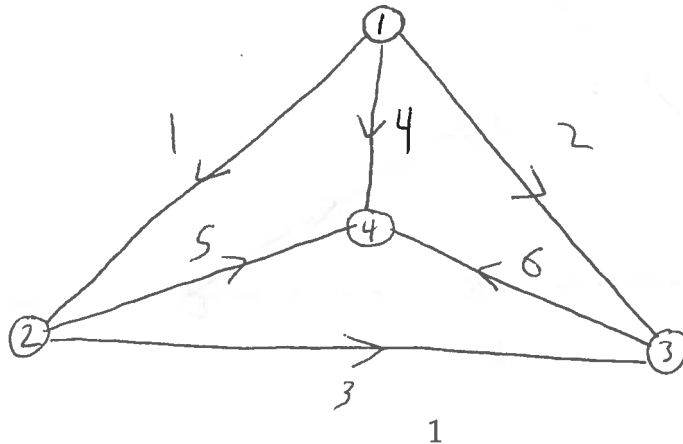
The incidence matrix for the graph is a matrix representation of the graph. Each row represents an edge, and each column represents a node. For a given row, there is a -1 if the edge is leaving the node, and a 1 if the edge is entering the node, and a 0 otherwise. The incidence matrix for the graph above is:

$$\begin{pmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{pmatrix}$$

If we perform elimination on this matrix we get

$$\begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The graph for this reduced incidence matrix is
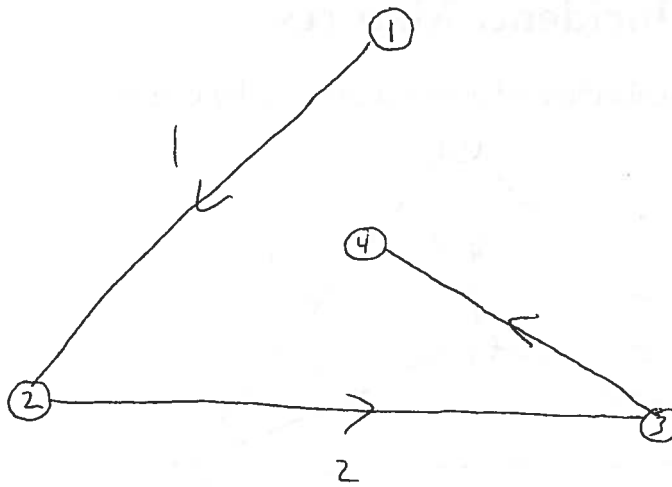
This graph is a *tree*. It has no loops. Once loops are introduced, the rows become dependent.

What are the four fundamental subspaces for incidence matrices?

Nullspace - The nullspace of an incidence matrix is spanned by the vector

$$\begin{pmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}.$$

The nullspace is the set of all solutions to the equation $A\mathbf{x} = \mathbf{0}$. If we view the graph as an electric circuit, we can view the input vector $\mathbf{x}$ as representing the potential (the voltate) at each node, and the output vector $A\mathbf{x}$ as representing the potential difference across each of the edges. If all potential differences are 0, then (assuming the graph is connected) all the voltages are the same.

Row Space - A vector $\mathbf{v}$ is in the rowspace of an incidence matrix if it's perpendicular to $(1, 1, 1, 1)$.

Example - Is the vector $(1, 2, 4, 3)$ in the rowspace of our incidence matrix? What about $(3, -2, 1, -2)$?

Column Space - The column space contains all linear combinations of the columns. How can we tell if a particular vector **b** is in the column space of an incidence matrix? The vector $A\mathbf{x}$ is a vector of differences. If we add differences around a closed loop in the graph, the cancellation leaves zero. If **b** is in the column space of A, then its components around any loop add up to zero. By testing each loop, we can tell if **b** is in the column space.

Left Nullspace - The left nullspace contains the solutions to $A^T\mathbf{y} = \mathbf{0}$. For example

$$
A^T\mathbf{y} = \begin{pmatrix} -1 & -1 & 0 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}
$$

What do the equations mean? Each represents the net flow into (or out of, depending on your point of view) a node. For circuits, these are Kirchoff's current laws. We find that every loop current is a solution to the current law, and that all such solutions are built from sums of loop currents.

Example - Show that the vector given by the flow around the larger triangle in the graph above is a linear combination of the flows around the smaller triangles.