

Math 2270 - Lecture 38 : Different Bases and the Pseudoinverse

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Fall 2012

This lecture covers *section 7.3* of the textbook.

Today, we're going to look a little deeper into this idea of representing the same linear transformations in different bases. Diagonalization, and the singular value decomposition, are really just ways of changing a linear transformation from the standard basis into the same linear transformation, but viewed from a basis where the computations are simpler.

We'll also revisit the four fundamental subspaces, and define something called the *pseudoinverse* of a matrix, which takes the place of the inverse when the inverse does not exist.

The assigned problems for this section are:

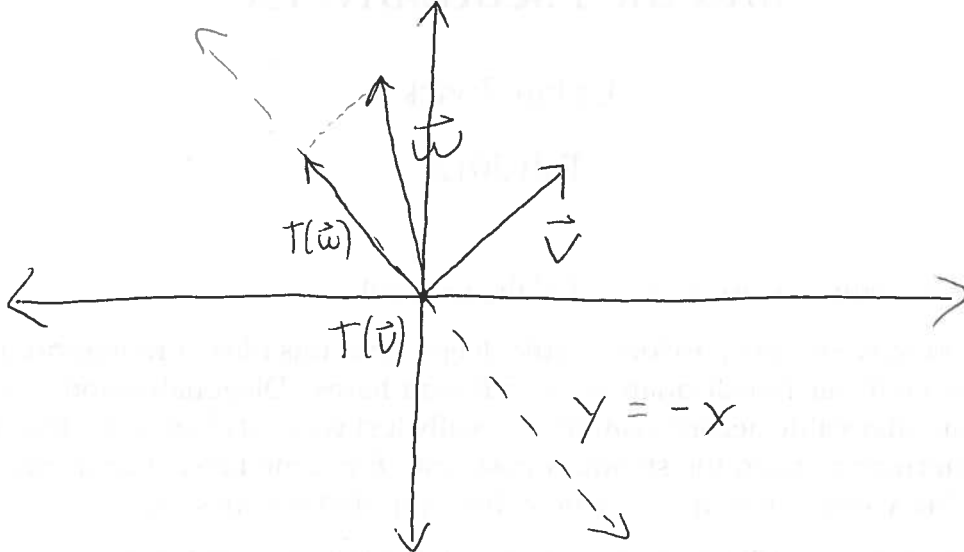
Section 7.3 - 1, 5, 6, 7, 9

Diagonalization and a Change of Basis

A linear transformation is a map, T , from a vector space V to a vector space U . In general there are many different choices of bases we can make for V and U , and each of these different bases will give us a different matrix representation for the linear transformation T . Note that in all of these the linear transformation T is the same, the only thing that changes is how we represent it.

Diagonalizing a matrix is really just finding a basis for which the linear transformation represented by that matrix is simplest. This basis is the basis of eigenvectors.

Let's see this with an example. Suppose we have the linear transformation that projects every vector onto the line $y = -x$:



In the standard basis this linear transformation is given by the matrix

$$A = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

The eigenvalues for this matrix are $\lambda_1 = 1$ and $\lambda_2 = 0$ with eigenvectors

$$\begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

If we use these vectors as our basis vectors, the linear transformation becomes

$$A' = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

If we have a vector represented in this eigenvector basis, and we want to represent the vector in the standard basis, the conversion matrix is

$$M = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

The inverse of this conversion matrix is

$$M^{-1} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

We can derive A' through the relation $A' = M^{-1}AM$:

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}.$$

What's going on here is that the matrices on the right are converting the vector to a vector in the standard basis, then taking the linear transformation, then converting the vector back into the eigenvector basis.

Let's do one more example. Suppose we have a basis consisting of the basis vectors

$$\begin{pmatrix} 2 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

We want the matrix A'' that represents our projection in this basis. What is it? Well, the matrix for converting from this new basis to the standard basis is

$$N = \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}.$$

The matrix for converting from the standard basis to this new basis is

$$N^{-1} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ 0 & 1 \end{pmatrix}.$$

So, the linear transformation in the new basis will be:

$$A'' = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}.$$

Note that all these matrices represent *the same* linear transformation, just in different bases.

Singular Value Decomposition

So far, we've assumed that the input and the output bases are always the same. This doesn't have to be the case. In fact, if we have a linear transformation $T : \mathbf{V} \rightarrow \mathbf{U}$ where \mathbf{V} and \mathbf{U} have different dimensions, it cannot be the case! We need something more general than diagonalization. We need the singular value decomposition.

What we're doing with the singular value decomposition is we're finding a basis $\mathbf{v}_1, \dots, \mathbf{v}_n$ for \mathbf{V} , and a basis $\mathbf{u}_1, \dots, \mathbf{u}_m$ for \mathbf{U} , for which the linear transformation T has a particularly nice form. Specifically, if the linear transformation T has rank r we choose basis vectors so that if A is the matrix representing the linear transformation we have the relations

$$A\mathbf{v}_j = \begin{cases} \sigma_j \mathbf{u}_j & j \leq r \\ \mathbf{0} & j > r \end{cases}$$

The Pseudoinverse

The singular value decomposition $A = U\Sigma V^T$ finds a set of basis vectors for \mathbf{V} and \mathbf{U} that allow us to represent the linear transformation A in a particularly nice way, namely, the matrix Σ .

Now, in general, Σ will look like

$$\Sigma = \begin{pmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_r \end{pmatrix}.$$

It won't, in general, be invertible. In fact, it won't, in general, even be square. However, we define the *pseudoinverse* of Σ as

$$\Sigma^+ = \begin{pmatrix} \sigma_1^{-1} & & \\ & \ddots & \\ & & \sigma_r^{-1} \end{pmatrix}.$$

where if Σ is $n \times m$, Σ^+ is $m \times n$. For example, if

$$\Sigma = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

then

$$\Sigma^+ = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and

$$\Sigma^+\Sigma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

which is as close to the identity as we can get with Σ .

We take this pseudoinverse of Σ to define the pseudoinverse of A :

$$A^+ = V\Sigma^+U^T$$

The pseudoinverse is as close to the inverse as you can get, in the following sense. If $Ax = \mathbf{b}$, then \mathbf{b} is a matrix in the column space of A , and $A^+\mathbf{b}$ will be the unique vector \mathbf{q} in the row space of A such that $A\mathbf{q} = \mathbf{b}$. If A is invertible, then $A^+ = A^{-1}$.

Example - Calculate the pseudoinverse of the matrix

$$A = \begin{pmatrix} 2 & 2 \\ 1 & 1 \end{pmatrix}$$

$$A^T = \begin{pmatrix} 2 & 1 \\ 2 & 1 \end{pmatrix} \quad A^T A = \begin{pmatrix} 2 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 2 & 2 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 5 & 5 \\ 5 & 5 \end{pmatrix}$$

$$\begin{vmatrix} 5-\lambda & 5 \\ 5 & 5-\lambda \end{vmatrix} = \lambda^2 - 10\lambda = \lambda(\lambda - 10)$$

$$\sigma_1 = \sqrt{10} \quad \sigma_2 = 0 \quad \Sigma = \begin{pmatrix} \sqrt{10} & 0 \\ 0 & 0 \end{pmatrix}$$

$$\lambda = 10 \quad \begin{pmatrix} -5 & 5 \\ 5 & -5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \vec{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ normalized } \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$\lambda = 0 \quad \begin{pmatrix} 5 & 5 \\ 5 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \vec{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \text{ normalized } \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$$

$$A \vec{v}_1 = \begin{pmatrix} 2 & 2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} 2\sqrt{2} \\ \sqrt{2} \end{pmatrix} \quad \frac{A \vec{v}_1}{\sigma_1} = \begin{pmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{pmatrix} = \vec{u}_1$$

$$A^T \vec{u}_2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 2 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \Rightarrow \vec{u}_2 = \begin{pmatrix} \frac{1}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}} \end{pmatrix}$$

$$A^+ = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{10}} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \end{pmatrix} = \begin{pmatrix} \frac{2}{10} & \frac{1}{10} \\ \frac{2}{10} & \frac{1}{10} \end{pmatrix} = \begin{pmatrix} \frac{1}{5} & \frac{1}{10} \\ \frac{1}{5} & \frac{1}{10} \end{pmatrix}$$