Math 2270 - Lecture 37 : Linear Transformations, Change of Bases, and Why Matrix Multiplication Is The Way It Is

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This lecture covers *section 7.2* of the textbook.

Today we're going to delve deeper into linear transformations, and their connections with matrices. We're also finally going to see why matrix multiplication works the way it does!

The assigned problems for this section are:

Section 7.2 - 5, 14, 15, 17, 26

Linear Transformation and Vector Spaces

We know that matrix multiplication represents a linear transformation, but can any linear transformation be represented by a matrix? The answer, it turns out, is yes. That's where matrices originated.

Thus far in our class we've restricted ourselves to dealing with finite-dimensional vector spaces over the real numbers. These restrictions aren't necessary for a vector space. We'll still maintain the assumption that our vector space is finite-dimensional, but besides that we'll deal with things in greater generality.

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Any finite-dimensional vector space $\mathbf{V}$ will have a basis. Suppose this basis is $\mathbf{v}_1, \ldots, \mathbf{v}_n$. Then any vector $\mathbf{v}$ in our vector space can be written as:

$$\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n.$$

Suppose we have another finite-dimensional vector space $\mathbf{w}$ with basis $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_m$, and a linear transformation, T, that takes vectors in $\mathbf{V}$ to vectors in $\mathbf{W}$. If we know what the transformation T does to the basis vectors of $\mathbf{V}$, then we know what it does to every other vector in $\mathbf{V}$. Read that last sentence again, because it's that important.

Suppose we know:

$$T(\mathbf{v}_1) = a_{11}\mathbf{w}_1 + a_{21}\mathbf{w}_2 + \cdots + a_{m1}\mathbf{w}_m$$
$$T(\mathbf{v}_2) = a_{12}\mathbf{w}_1 + a_{22}\mathbf{w}_2 + \cdots + a_{m2}\mathbf{w}_m$$
$$\vdots$$
$$T(\mathbf{v}_n) = a_{1n}\mathbf{w}_1 + a_{2n}\mathbf{w}_2 + \cdots + a_{mn}\mathbf{w}_m.$$

Then if $\mathbf{v}$ is *any* vector in $\mathbf{V}$, we can write it in terms of the basis $\mathbf{v}_1, \ldots, \mathbf{v}_n$:

$$\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n.$$

The action of T on $\mathbf{v}$ will then be given by

$$T(\mathbf{v}) = c_1 T(\mathbf{v}_1) + c_2 T(\mathbf{v}_2) + \cdots + c_n T(\mathbf{v}_n).$$

Note how important the idea of linearity is here! In fact, we can represent $\mathbf{v}$ compactly as

$$\mathbf{v} = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}$$

And the action of T on **v** will just be:

$$T(\mathbf{v}) = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}.$$

That's right, the linear transformation has an associated matrix!

Any linear transformation from a finite dimension vector space **V** with dimension n to another finite dimensional vector space **W** with dimension m can be represented by a matrix. This is why we study matrices.

Example - Suppose we have a linear transformation T taking **V** to **W**, where both **V** and **W** are 2-dimensional vector spaces. If the action of this transformation on the basis vectors of **V** is:

$$T(\mathbf{v}_1) = 2\mathbf{w}_1 + 3\mathbf{w}_2,$$

$$T(\mathbf{v}_2) = 3\mathbf{w}_1 + 1\mathbf{w}_2,$$

what is the matrix representing this transformation? Also, what is $T(\mathbf{v})$, written in terms of the basis vector of **W**, if

$$\mathbf{v} = 1\mathbf{v}_1 + 4\mathbf{v}_2?$$

$$A = \begin{pmatrix} 2 & 3 \\ 3 & 1 \end{pmatrix} \qquad \begin{pmatrix} 2 & 3 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 4 \end{pmatrix} = \begin{pmatrix} 14 \\ 7 \end{pmatrix}$$

$$T(\vec{v}) = T(\vec{v}_1) + 4\,T(v_2)$$
$$= 2\vec{\omega}_1 + 3\vec{\omega}_2 + 4(3\vec{\omega}_1 + \vec{\omega}_2) = 14\vec{\omega}_1 + 7\vec{\omega}_2.$$

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Matrix Multiplication

Suppose we have a linear transformation S from a 2-dimensional vector space $\mathbf{U}$, to another 2-dimension vector space $\mathbf{V}$, and then another linear transformation T from $\mathbf{V}$ to another 2-dimensional vector space $\mathbf{W}$. Suppose we have a vector $\mathbf{u} \in \mathbf{U}$:

$$\mathbf{u} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2.$$

Suppose S maps the basis vectors of $\mathbf{U}$ as follows:

$$S(\mathbf{u}_1) = a_{11}\mathbf{v}_1 + a_{21}\mathbf{v}_2,$$

$$S(\mathbf{u}_2) = a_{12}\mathbf{v}_1 + a_{22}\mathbf{v}_2.$$

Then the action of S on $\mathbf{u}$ will be:

$$S(\mathbf{u}) = c_1(a_{11}\mathbf{v}_1 + a_{21}\mathbf{v}_2) + c_2(a_{12}\mathbf{v}_1 + a_{22}\mathbf{v}_2) = (a_{11}c_1 + a_{12}c_2)\mathbf{v}_1 + (a_{21}c_1 + a_{22}c_2)\mathbf{v}_2.$$

Now, suppose T maps the basis vectors of $\mathbf{V}$ as follows:

$$T(\mathbf{v}_1) = b_{11}\mathbf{w}_1 + b_{21}\mathbf{w}_2,$$

$$T(\mathbf{v}_2) = b_{12}\mathbf{w}_1 + b_{22}\mathbf{w}_2.$$

Then the action of T on $S(\mathbf{u})$ will be:

$$T(S(\mathbf{u})) = (a_{11}c_1 + a_{12}c_2)(b_{11}\mathbf{w}_1 + b_{21}\mathbf{w}_2) + (a_{21}c_1 + a_{22}c_2)(b_{12}\mathbf{w}_1 + b_{22}\mathbf{w}_2) = ((a_{11}b_{11} + a_{21}b_{12})c_1 + (a_{12}b_{11} + a_{22}b_{12})c_2)\mathbf{w}_1 + ((a_{11}b_{21} + a_{21}b_{22})c_1 + (a_{12}b_{21} + a_{22}b_{22})c_2)\mathbf{w}_2.$$

Or, in matrix form:

$$\begin{pmatrix} (a_{11}b_{11} + a_{21}b_{12})c_1 + (a_{12}b_{11} + a_{22}b_{12})c_2 \\ (a_{11}b_{21} + a_{21}b_{22})c_1 + (a_{12}b_{21} + a_{22}b_{22})c_2 \end{pmatrix}$$

Clear as mud? Well, we can rewrite this as:

$$\left(\begin{array}{cc} b_{11}a_{11} + b_{12}a_{21} & b_{11}a_{12} + b_{12}a_{22} \\ b_{21}a_{11} + b_{22}a_{21} & b_{21}a_{12} + b_{22}a_{22} \end{array} \right) \left(\begin{array}{c} c_1 \\ c_2 \end{array} \right).$$

Looking more familiar? How about now:

$$\left(\begin{array}{cc} b_{11} & b_{12} \\ b_{21} & b_{22} \end{array} \right) \left(\begin{array}{cc} a_{11} & a_{12} \\ a_{21} & a_{22} \end{array} \right) \left(\begin{array}{c} c_1 \\ c_2 \end{array} \right).$$

That's right, it's just matrix multiplication! We finally see the reason we multiply matrices in that weird way. Matrix multiplication corresponds to composition of linear transformations. At last it makes some sense![1]

Linear Transformations in Different Bases

Suppose we have two bases for $\mathbb{R}^2$: the standard basis, and another basis given by

$$\left(\begin{array}{c} 2 \\ 1 \end{array} \right), \left(\begin{array}{c} 1 \\ 1 \end{array} \right).$$

To transform a vector written in terms of the second basis into a vector written in terms of the standard basis, we multiply it by the *basis change matrix*:

$$M = \left(\begin{array}{cc} 2 & 1 \\ 1 & 1 \end{array} \right)$$

To go the other way, taking a vector written in terms of the standard basis and writing it in terms of the second basis, we would multiply by the inverse of the basis change matrix:

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[1]Note this is true for composition of linear transformations for any size vector spaces, I just chose 2×2 because it's relatively easy.

$$M^{-1} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}.$$

Suppose we have a linear transformation T. A is the matrix that represents this transformation in the standard basis, while B is the matrix representing this transformation in the second basis. How are these matrices related? Well, Suppose we're given A, and a vector $\mathbf{v}$ represented in terms of the second basis. To find the action of T on $\mathbf{v}$ we'd first multiply $\mathbf{v}$ by M to get it in terms of the standard basis, and then multiply $M\mathbf{v}$ by A to get the transformation. This would give us the transformed vector, but in terms of the standard basis. To get this transformed vector in terms of the second basis we'd multiply $AM\mathbf{v}$ by M^{-1}. So, the matrix for our transformation is $B = M^{-1}AM$.

Look familiar? It should. If two matrices represent the same transformation, just in different bases, then the matrices are *similar*. That's what's so important about similar matrices, and why mathematicians are interested in studying them. If you want to understand linear transformations and don't want to have to tie them to certain bases, you want to understand the equivalence class of similar matrices.

Example - If the linear transformation T is represented in the second basis above as:

$$\begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix}$$

what is its representation in the standard basis?

$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} -1 & 4 \\ -3 & 7 \end{pmatrix} = \begin{pmatrix} -5 & 15 \\ -4 & 11 \end{pmatrix}$$