Math 2270 - Lecture 36 : The Idea of a Linear Transformation

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This lecture covers *section 7.1* of the textbook.

Today's lecture will be a bit of a break from the relatively high level of difficulty of the last few lectures. Today, we're going to talk about a fundamental idea in mathematics, that of a linear tranformation. Linear transformations are deeply, intimately connected to matrices. In fact, for linear transformations from $\mathbb{R}^n$ to $\mathbb{R}^m$, matrices and linear transformations are in bijective correspondence. There's a reason we call it linear algebrea, after all!

The assigned problems for this section are:

Section 7.1 - 1, 3, 4, 10, 16,

1 The Basics

A linear transformation is a function on a vector space that assigns to each input vector $\mathbf{v}$ and output vector $T(\mathbf{v})$, and satisfies the following requirements

- $T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$,
- $T(c\mathbf{v}) = cT(\mathbf{v})$.

Note that $\mathbf{v}$ and $T(\mathbf{v})$ may be in different vector spaces.

We can combine these two definitions and say a transformation T is linear if it satisfies

$$T(c\mathbf{u} + d\mathbf{v}) = cT(\mathbf{u}) + dT(\mathbf{v}),$$

where c, d are scalars and $\mathbf{u}, \mathbf{v}$ are vectors.

Note that translation $T(\mathbf{v}) = \mathbf{v} + \mathbf{u}_0$ for a fixed $\mathbf{u}_0$ is not a linear transformation if $\mathbf{u}_0 \neq 0$. This is because

$$T(\mathbf{v} + \mathbf{w}) = \mathbf{v} + \mathbf{w} + \mathbf{u}_0$$

while

$$T(\mathbf{v}) + T(\mathbf{w}) = \mathbf{v} + \mathbf{u}_0 + \mathbf{w} + \mathbf{u}_0.$$

Example - Is the transformation $T(v_1, v_2) = v_1 v_2$ a linear transformation?

No, $T(1,1) = 1$ $(1,0) + (0,1) = (1,1)$

But, $T(1,0) + T(0,1) = 0 + 0 = 0 \neq 1$

Now, linear transformations take lines to lines, and triangles to triangles. What I mean is that if three points p_1, p_2, p_3 are collinear, then $T(p_1), T(p_2), T(p_3)$ are too.

Note that if A is a matrix then the transformation $T(\mathbf{x}) = A\mathbf{x}$ is a linear transformation.

Example - Is projection onto the xy-plane a linear transformation?

Yes. $T(x, y, z) = (x, y, 0)$.

$$T(x_1+x_2, y_1+y_2, z_1+z_2) = (x_1+x_2, y_1+y_2, 0)$$
$$= T(x_1, y_1, z_1) + T(x_2, y_2, z_2) \checkmark$$

$$T(cx, cy, cz) = (cx, cy, 0) = c(x, y, 0)$$
$$= c\,T(x, y, z). \checkmark$$

Example - What about projection onto the plane $z = 1$?

No, For example

$$T(1,1,1) = (1,1,1)$$

$$T(2,2,2) = (2,2,1) \neq T(1,1,1) + T(1,1,1).$$