

Math 2270 - Lecture 36 : The Idea of a Linear Transformation

Dylan Zwick

Fall 2012

This lecture covers *section 7.1* of the textbook.

Today's lecture will be a bit of a break from the relatively high level of difficulty of the last few lectures. Today, we're going to talk about a fundamental idea in mathematics, that of a linear transformation. Linear transformations are deeply, intimately connected to matrices. In fact, for linear transformations from \mathbb{R}^n to \mathbb{R}^m , matrices and linear transformations are in bijective correspondence. There's a reason we call it linear algebra, after all!

The assigned problems for this section are:

Section 7.1 - 1, 3, 4, 10, 16,

1 The Basics

A linear transformation is a function on a vector space that assigns to each input vector \mathbf{v} and output vector $T(\mathbf{v})$, and satisfies the following requirements

- $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$,
- $T(c\mathbf{v}) = cT(\mathbf{v})$.

Note that \mathbf{v} and $T(\mathbf{v})$ may be in different vector spaces.

We can combine these two definitions and say a transformation T is linear if it satisfies

$$T(c\mathbf{u} + d\mathbf{v}) = cT(\mathbf{u}) + dT(\mathbf{v}),$$

where c, d are scalars and \mathbf{u}, \mathbf{v} are vectors.

Note that translation $T(\mathbf{v}) = \mathbf{v} + \mathbf{u}_0$ for a fixed \mathbf{u}_0 is not a linear transformation if $\mathbf{u}_0 \neq 0$. This is because

$$T(\mathbf{v} + \mathbf{w}) = \mathbf{v} + \mathbf{w} + \mathbf{u}_0$$

while

$$T(\mathbf{v}) + T(\mathbf{w}) = \mathbf{v} + \mathbf{u}_0 + \mathbf{w} + \mathbf{u}_0.$$

Example - Is the transformation $T(v_1, v_2) = v_1v_2$ a linear transformation?

Now, linear transformations take lines to lines, and triangles to triangles. What I mean is that if three points p_1, p_2, p_3 are collinear, then $T(p_1), T(p_2), T(p_3)$ are too.

Note that if A is a matrix then the transformation $T(\mathbf{x}) = A\mathbf{x}$ is a linear transformation.

Example - Is projection onto the xy -plane a linear transformation?

Example - What about projection onto the plane $z = 1$?