Math 2270 - Lecture 35 : Singular Value Decomposition (SVD)

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This lecture covers *section 6.7* of the textbook.

Today, we summit diagonal mountain. That is to say, we'll learn about the most general way to "diagonalize" a matrix. This is called the *singular value decomposition*. It's kind of a big deal.

Up to this point in the chapter we've dealt exclusively with square matrices. Well, today, we're going to allow rectangular matrices. Is A is an $m \times n$ matrix with $m \neq n$ then the eigenvalue equation

$$A\mathbf{x} = \lambda\mathbf{x}$$

has issues. In particular, the vector $\mathbf{x}$ will have n components, while the vector $A\mathbf{x}$ will have m components (!) and so the equation above won't make sense.

Well... nuts. Now what do we do? We need a square matrix. Well, as we learned when we were learning about projections, the matrices A^TA and AA^T will be square. They will also be symmetric, and in fact positive semidefinite. A diagonalizer's dream!

Making use of AA^T and A^TA, we'll construct the singular value decomposition of A.

The assigned problems for this section are:

Section 6.7 - 1, 4, 6, 7, 9.

1 The Singular Value Decomposition

Suppose A is an $m \times n$ matrix with rank r. The matrix AA^T will be $m \times m$ and have rank r. The matrix A^TA will be $n \times n$ and also have rank r. Both matrices A^TA and AA^T will be positive semidefinite, and will therefore have r (possibly repeated) positive eigenvalues, and r linearly independent corresponding eigenvectors. As the matrices are symmetric, these eigenvectors will be orthogonal, and we can choose them to be orthonormal.

We call the eigenvectors of A^TA corresponding to its non-zero eigenvalues $\mathbf{v}_1, \ldots, \mathbf{v}_r$. These vectors will be in the row space of A. We call the eigenvectors of AA^T corresponding to its non-zero eigenvalues $\mathbf{u}_1, \ldots, \mathbf{u}_r$. These vectors will be in the column space of A.

Now, these vectors have a remarkable relation. Namely,

$$A\mathbf{v}_1 = \sigma_1\mathbf{u}_1, A\mathbf{v}_2 = \sigma_2\mathbf{u}_2, \ldots, A\mathbf{v}_r = \sigma_r\mathbf{u}_r$$

where $\sigma_1, \ldots, \sigma_r$ are positive numbers called the *singular values* of the matrix A.

This relation lets us write

$$A\begin{pmatrix} \mathbf{v}_1 & \cdots & \mathbf{v}_r \end{pmatrix} = \begin{pmatrix} \mathbf{u}_1 & \cdots & \mathbf{u}_r \end{pmatrix}\begin{pmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_r \end{pmatrix}.$$

This gives us a decomposition $AV = U\Sigma$.

Noting that the columns of V are orthonormal we can right multiply both sides of this equality by V^T to get $A = U\Sigma V^T$. This is the singular value decomposition of A.

If we want to we can make V and U square. We just append orthonormal vectors $\mathbf{v}_{r+1}, \ldots, \mathbf{v}_n$ in the nullspace of A to V, and orthonormal vectors $\mathbf{u}_{r+1}, \ldots, \mathbf{u}_m$ in the left-nullspace of A to M. We'll still get $AV = U\Sigma$ and $A = U\Sigma V^T$.

This singular value decomposition has a particularly nice representation if we carry through the multiplication of the matrices:

$$A = U\Sigma V^T = \mathbf{u}_1\sigma_1\mathbf{v}_1 + \cdots + \mathbf{u}_r\sigma_r\mathbf{v}_r^T.$$

Each of these "pieces" has rank 1. If we order the singular values

$$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r$$

then the singular value decomposition gives A in r rank 1 pieces in *order of importance*.

We should prove the singular value decomposition before we compute some examples.

Proof of the Singular Value Decomposition - The matrices A^TA and AA^T, as we learned in section 6.5, are positive semidefinite. Therefore, all non-zero eigenvalues will be positive.

If λ_i is a non-zero eigenvalue of A^TA with eigenvector $\mathbf{v}_i$ then we can write $A^TA\mathbf{v}_i = \sigma_i^2\mathbf{v}_i$, where $\sigma_i = \sqrt{\lambda_i}$ is the positive square root of λ_i.

If we left multiply $A^TA\mathbf{v}_i = \sigma_i^2\mathbf{v}_i$ by $\mathbf{v}_i^T$ we get

$$\mathbf{v}_i^TA^TA\mathbf{v}_i = \sigma_i^2\mathbf{v}_i^T\mathbf{v}_i,$$

and therefore

$$\mathbf{v}_i^TA^TA\mathbf{v}_i = (A\mathbf{v}_i)^T(A\mathbf{v}_i) = ||A\mathbf{v}_i||^2 = \sigma_i^2\mathbf{v}_i^T\mathbf{v}_i = \sigma_i^2.$$

The last equality uses that $\mathbf{v}_i$ is normalized. So, this gives us $||A\mathbf{v}_i|| = \sigma_i$.

Now, as $A^TA\mathbf{v}_i = \sigma_i^2A\mathbf{v}_i$ if we left multiply both sides of this equation by A we get

$$AA^TA\mathbf{v}_i = \sigma_i^2A\mathbf{v}_i$$

and so $A\mathbf{v}_i$ is an eigenvector of AA^T, with eigenvalue σ_i^2. So, $\mathbf{u}_i = A\mathbf{v}_i/\sigma_i$ is a unit eigenvector of AA^T, and we have

$$A\mathbf{v}_i = \sigma_i \mathbf{u}_i.$$

Done!

2 Finding Singular Value Decompositions

Let's calculate a few singular value decompositions. First, let's start with the rank 2 unsymmetric matrix

$$A = \begin{pmatrix} 2 & 2 \\ -1 & 1 \end{pmatrix}.$$

A is not symmetric, and there will be no orthogonal matrix Q that will make $Q^{-1}AQ$ diagonal. We need two different orthogonal matrices U and V.

We find these matrices with the singular value decomposition. So, we want to compute A^TA and its eigenvectors.

$$A^TA = \begin{pmatrix} 5 & 3 \\ 3 & 5 \end{pmatrix}$$

and so

$$\begin{vmatrix} 5-\lambda & 3 \\ 3 & 5-\lambda \end{vmatrix} = (5-\lambda)^2 - 9 = \lambda^2 - 10\lambda + 16 = (\lambda-8)(\lambda-2).$$

So, A^TA has eigenvalues 8 and 2. The corresponding eigenvectors will be

$$\mathbf{v}_1 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \qquad \mathbf{v}_2 = \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}.$$

Now, to find the vectors $\mathbf{u}_1$ and $\mathbf{u}_2$ we multiply $\mathbf{v}_1$ and $\mathbf{v}_2$ by A:

$$A\mathbf{v}_1 = \begin{pmatrix} 2 & 2 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} 2\sqrt{2} \\ 0 \end{pmatrix},$$

$$A\mathbf{v}_2 = \begin{pmatrix} 2 & 2 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} 0 \\ \sqrt{2} \end{pmatrix}.$$

So, the unit vectors $\mathbf{u}_1$ and $\mathbf{u}_2$ will be:

$$\mathbf{u}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \mathbf{u}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The singular values will be $2\sqrt{2} = \sqrt{8}$ and $\sqrt{2}$. This gives us the singular value decomposition:

$$\begin{pmatrix} 2 & 2 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2\sqrt{2} & 0 \\ 0 & \sqrt{2} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}.$$

Example - Find the SVD of the matrix

$$A = \begin{pmatrix} 2 & 2 \\ 1 & 1 \end{pmatrix}.$$

$$A^+ = \begin{pmatrix} 2 & 1 \\ 2 & 1 \end{pmatrix} \quad A^+A = \begin{pmatrix} 2 & 1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 2 & 2 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 5 & 5 \\ 5 & 5 \end{pmatrix}$$

$$\begin{vmatrix} 5-\lambda & 5 \\ 5 & 5-\lambda \end{vmatrix} = (5-\lambda)^2 - 25 = \lambda^2 - 10\lambda = \lambda(\lambda - 10)$$

$$\lambda = 0, 10$$

$$\lambda = 10 \quad \begin{pmatrix} -5 & 5 \\ 5 & -5 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \vec{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \vec{v}_1 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$\begin{pmatrix} 2 & 2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} 2\sqrt{2} \\ \sqrt{2} \end{pmatrix} \Rightarrow \vec{u}_1 = \frac{1}{\sqrt{10}}\begin{pmatrix} 2\sqrt{2} \\ \sqrt{2} \end{pmatrix} = \begin{pmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{pmatrix}$$

$$\boxed{\begin{pmatrix} 2 & 2 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{pmatrix} \begin{pmatrix} \sqrt{10} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}}$$