

Math 2270 - Lecture 34 : Similar Matrices

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Fall 2012

This lecture covers *section 6.6* of the textbook.

Today we're going to talk about a relation between two matrices known as *similarity*. If a matrix is diagonalizable we can write it as

$$A = S^{-1}\Lambda S,$$

where Λ is the matrix of the eigenvalues of A , and the corresponding eigenvectors are the columns of S .

This is really useful... when it can be done. However, it is not the case that all matrices are diagonalizable. What we want to do today is to introduce a generalization of the notion of diagonalizing a matrix that works for *all* matrices.

The assigned problems for this section are:

Section 6.6 - 3, 9, 10, 12, 13

1 Similar Matrices

Two matrices A and B are *similar* if there exists an invertible matrix M such that

$$B = M^{-1}AM.$$

Note that, obviously, A is similar to itself, and if $B = M^{-1}AM$ then $A = MBM^{-1}$, so if A is similar to B , then B is similar to A . Finally, note that if $A = M_1^{-1}BM_1$ and $B = M_2^{-1}CM_2$ then $A = M_1^{-1}(M_2^{-1}CM_2)M_1 = (M_2M_1)^{-1}C(M_2M_1)$. So, if A is similar to B , and B is similar to C , then A is similar to C . In mathematics we say the relation “similar” is reflexive, symmetric, and transitive, and is therefore an *equivalence relation*.

A type of problem that comes up all the time in mathematics is you’re given a set of objects, and an equivalence relation on these objects, and you’re asked to classify all the objects up to this equivalence relation. These are known as classification problems, and they’re in general *very hard*.

Today, we’re going to talk about this problem in the context of matrices. More precisely, we’re going to classify all matrices up to similarity. That’s right, by the end of the day, you’ll have seen a completed classification problem.

2 Eigenvalues and Eigenvectors of Similar Matrices

Two similar matrices have *the same* eigenvalues, even though they will usually have different eigenvectors. Said more precisely, if $B = M^{-1}AM$ and \mathbf{x} is an eigenvector of A , then $M^{-1}\mathbf{x}$ is an eigenvector of $B = M^{-1}AM$.

The proof is quick. Suppose $A\mathbf{x} = \lambda\mathbf{x}$. Then as $A = MBM^{-1}$ we have

$$MBM^{-1}\mathbf{x} = \lambda\mathbf{x} \Rightarrow B(M^{-1}\mathbf{x}) = \lambda(M^{-1}\mathbf{x}).$$

So, $M^{-1}\mathbf{x}$ is an eigenvector for B , with eigenvalue λ . Note that as M is invertible so is, of course, M^{-1} , and so M^{-1} has full rank. Having full rank means its columns are linearly independent, and so if $\mathbf{x} \neq \mathbf{0}$ we must have $M^{-1}\mathbf{x} \neq \mathbf{0}$.

We see right away that if two matrices have different eigenvalues then they are not similar. Also, if two matrices have the same *distinct* eigenvalues then they are similar. Suppose A and B have the same distinct eigenvalues. Then they are both diagonalizable with the same diagonal

matrix Λ . So, both A and B are similar to Λ , and therefore A is similar to B .

However, if two matrices have the same *repeated* eigenvalues they may not be distinct. For example, the zero matrix

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

has the repeated eigenvalue 0, but is only similar to itself. On the other hand the matrix

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

also has the repeated eigenvalue 0, but is not similar to the 0 matrix. It is similar to every matrix of the form

$$\begin{pmatrix} cd & d^2 \\ -c^2 & -cd \end{pmatrix}$$

besides the zero matrix.

Example - Prove the above statement.

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad M^{-1} = \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \quad \text{Pick } a, b \text{ so } ad-bc \neq 0$$

$$\begin{aligned} M^{-1}AM &= \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \\ &= \begin{pmatrix} 0 & d \\ 0 & -c \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \\ &= \begin{pmatrix} cd & d^2 \\ -c^2 & -cd \end{pmatrix} \quad \checkmark \end{aligned}$$

Note: If $c, d = 0$ then $ad-bc = 0$.

3 The Jordan Form

Not only can the matrix

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

not be diagonalized, but it's as close to diagonal as possible. It's in something called *Jordan form*. A *Jordan matrix* is a matrix of the form

$$\begin{pmatrix} \lambda & 1 & & \\ & \lambda & 1 & \\ & & \ddots & \\ & & & \lambda \end{pmatrix}.$$

So, for example, the matrix

$$\begin{pmatrix} 5 & 1 & 0 \\ 0 & 5 & 1 \\ 0 & 0 & 5 \end{pmatrix}$$

is a Jordan matrix. Every Jordan matrix has only one eigenvalue, the number on the diagonal, and only one independent eigenvector.

Now, if a matrix A has s independent eigenvectors, then it is similar to a matrix J that has s Jordan blocks on its diagonal:

$$M^{-1}AM = \begin{pmatrix} J_1 & & \\ & \ddots & \\ & & J_s \end{pmatrix}.$$

These Jordan blocks are Jordan matrices, and the diagonal terms on these Jordan blocks correspond to the eigenvalues of the eigenvectors. The matrix above is called the *Jordan form* of A , and it is unique up to permuting the Jordan blocks.

This completely classifies *all* matrices up to similarity. Pretty neat, huh?