Math 2270 - Lecture 32 : Symmetric Matrices

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This lecture covers *section 6.4* of the textbook.

Today we're going to look at diagonalizing a matrix when the matrix is symmetric. It turns out that symmetric matrices have a number of totally awesome properties:

1. The eigenvalues of a symmetric matrix are all *real*.
2. The eigenvectors of a symmetrix matrix are all orthogonal, and hence can be chosen orthonormal.

Today we'll prove these properties, and a few more, and learn yet another factorization, this one being $A = Q\Lambda Q^T$, combining diagonalization with orthogonal matrices.

The assigned problems for this section are:

Section 6.4 - 1, 3, 5, 14, 23

1 Eigenvectors and Eigenvalues of Symmetric Matrices

We'll begin by proving that all the eigenvalues of a symmetric matrix are real. First, recall that the complex conjugate of an imaginary number

$$z = x + iy$$

is the number you get by switching the sign of the imaginary term, and is usually denoted with a line over the variable

$$\overline{z} = x - iy.$$

Now, the complex conjugate of a product is the product of the conjugate $\overline{xy} = \overline{x}\overline{y}$, and so if we take the eigenvalue equation for a real, symmetric matrix A we get

$$A\mathbf{x} = \lambda\mathbf{x}$$

and conjugate both sides we get

$$A\overline{\mathbf{x}} = \overline{\lambda}\overline{\mathbf{x}}.$$

Note that $\overline{A} = A$, as A is real.

If we take the transpose of the conjugate equation we get (remembering that A is symmetric, so $A = A^T$)

$$\overline{\mathbf{x}}^T A = \overline{\mathbf{x}}^T\overline{\lambda}.$$

If we right multiply the above equation by $\mathbf{x}$ we get

$$\overline{\mathbf{x}}^T A\mathbf{x} = \overline{\mathbf{x}}\overline{\lambda}\mathbf{x}.$$

If we left multiply our original eigenvalue equation by $\overline{\mathbf{x}}^T$ we get

$$\overline{\mathbf{x}}^T A\mathbf{x} = \overline{\mathbf{x}}^T\lambda\mathbf{x}.$$

We note that both of these are equalities for $\overline{\mathbf{x}}^T A\mathbf{x}$, and therefore we must have the equality

$$\overline{\mathbf{x}}^T \lambda \mathbf{x} = \overline{\mathbf{x}}^T \overline{\lambda} \mathbf{x}$$

So, we must have $\lambda = \overline{\lambda}$. Thus, λ must be real.

Now we prove that all the eigenvectors are orthogonal. We first prove that if the eigenvalues are different, the eigenvectors must be orthogonal.

Suppose $A\mathbf{x} = \lambda_1 \mathbf{x}$ and $A\mathbf{y} = \lambda_2 \mathbf{y}$. From these we get

$$(\lambda_1 \mathbf{x})^T \mathbf{y} = (A\mathbf{x})^T \mathbf{y} = \mathbf{x} A^T \mathbf{y} = \mathbf{x}^T A \mathbf{y} = \mathbf{x}^T \lambda_2 \mathbf{y}.$$

The only way this is true is if $\lambda_1 = \lambda_2$ or $\mathbf{x}^T \mathbf{y} = 0$. So, if the eigenvalues are distinct, the eigenvectors are orthogonal.

We have now proven that if all the eigenvalues of a matrix are distinct, then all the eigenvectors are perpendicular. We can scale the eigenvectors so that they all have length 1, and are therefore orthonormal.

So, if A is symmetric, we can diagonalize it as

$$A = Q\Lambda Q^T = \begin{pmatrix} \mathbf{x}_1 & \cdots & \mathbf{x}_n \end{pmatrix} \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} \begin{pmatrix} \mathbf{x}_1^T \\ \vdots \\ \mathbf{x}_n^T \end{pmatrix}$$

Note this can be rewritten as

$$A = \lambda_1 \mathbf{x}_1 \mathbf{x}_1^T + \cdots + \lambda_n \mathbf{x}_n^T \mathbf{x}_n$$

where the $\mathbf{x}_i \mathbf{x}_i^T$ terms are the projection matrices onto the linear spaces spanned by the respective $\mathbf{x}_i$. These are called the *eigenspaces*.

The book proves at the end of the section that this is true for *any* symmetric matrix, regardless of whether the eigenvalues are repeated or not. So, if A is a symmetric matrix, then A is diagonalizable, and can be written as

$$A = Q\Lambda Q^{-1} = Q\Lambda Q^T.$$

This is known as the *spectral theorem*. Note that, as we mentioned in the last lecture, not all matrices are diagonalizable. But, all symmetric matrices are.

Example - Find the spectral decomposition $A = Q\Lambda Q^T$ of the matrix

$$A = \begin{pmatrix} -2 & 6 \\ 6 & 7 \end{pmatrix}$$

$$\begin{vmatrix} -2-\lambda & 6 \\ 6 & 7-\lambda \end{vmatrix} = (-2-\lambda)(7-\lambda) - 36$$

$$= \lambda^2 - 5\lambda - 50 = (\lambda - 10)(\lambda + 5)$$

Eigenvalues $\lambda = 10, -5$

$\lambda = 10$ $\quad \begin{pmatrix} -12 & 6 \\ 6 & -3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \Rightarrow \vec{x}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad \hat{x}_1 = \begin{pmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{pmatrix}$

$\lambda = -5$ $\quad \begin{pmatrix} 3 & 6 \\ 6 & 12 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \Rightarrow \vec{x}_2 = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \quad \hat{x}_2 = \begin{pmatrix} \frac{2}{\sqrt{5}} \\ -\frac{1}{\sqrt{5}} \end{pmatrix}$

$$\begin{pmatrix} -2 & 6 \\ 6 & 7 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \end{pmatrix}\begin{pmatrix} 10 & 0 \\ 0 & -5 \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \end{pmatrix}$$

2 Complex Eigenvalues of Real Matrices

If a real matrix is symmetric then all its eigenvalues are real. However, if a real matrix is not symmetric, it's quite possible that there are complex eigenvalues or even complex eigenvectors. However, these eigenvalues and eigenvectors will come in "conjugate pairs"

$$A\mathbf{x} = \lambda\mathbf{x} \qquad\qquad A\overline{\mathbf{x}} = \overline{\lambda}\overline{\mathbf{x}}$$

So, each complex eigenvalue has a conjugate twin.