

Math 2270 - Lecture 32 : Symmetric Matrices

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This lecture covers *section 6.4* of the textbook.

Today we're going to look at diagonalizing a matrix when the matrix is symmetric. It turns out that symmetric matrices have a number of totally awesome properties:

1. The eigenvalues of a symmetric matrix are all *real*.
2. The eigenvectors of a symmetric matrix are all orthogonal, and hence can be chosen orthonormal.

Today we'll prove these properties, and a few more, and learn yet another factorization, this one being $A = Q\Lambda Q^T$, combining diagonalization with orthogonal matrices.

The assigned problems for this section are:

Section 6.4 - 1, 3, 5, 14, 23

1 Eigenvectors and Eigenvalues of Symmetric Matrices

We'll begin by proving that all the eigenvalues of a symmetric matrix are real. First, recall that the complex conjugate of an imaginary number

$$z = x + iy$$

is the number you get by switching the sign of the imaginary term, and is usually denoted with a line over the variable

$$\bar{z} = x - iy.$$

Now, the complex conjugate of a product is the product of the conjugate $\overline{xy} = \bar{x}\bar{y}$, and so if we take the eigenvalue equation for a real, symmetric matrix A we get

$$A\mathbf{x} = \lambda\mathbf{x}$$

and conjugate both sides we get

$$A\bar{\mathbf{x}} = \bar{\lambda}\bar{\mathbf{x}}.$$

Note that $\bar{A} = A$, as A is real.

If we take the transpose of the conjugate equation we get (remembering that A is symmetric, so $A = A^T$)

$$\bar{\mathbf{x}}^T A = \bar{\mathbf{x}}^T \bar{\lambda}.$$

If we right multiply the above equation by \mathbf{x} we get

$$\bar{\mathbf{x}}^T A\mathbf{x} = \bar{\mathbf{x}}^T \bar{\lambda}\mathbf{x}.$$

If we left multiply our original eigenvalue equation by $\bar{\mathbf{x}}^T$ we get

$$\bar{\mathbf{x}}^T A\mathbf{x} = \bar{\mathbf{x}}^T \lambda\mathbf{x}.$$

We note that both of these are equalities for $\bar{\mathbf{x}}^T A\mathbf{x}$, and therefore we must have the equality

$$\bar{\mathbf{x}}^T \lambda \mathbf{x} = \bar{\mathbf{x}}^T \bar{\lambda} \mathbf{x}$$

So, we must have $\lambda = \bar{\lambda}$. Thus, λ must be real.

Now we prove that all the eigenvectors are orthogonal. We first prove that if the eigenvalues are different, the eigenvectors must be orthogonal.

Suppose $A\mathbf{x} = \lambda_1\mathbf{x}$ and $A\mathbf{y} = \lambda_2\mathbf{y}$. From these we get

$$(\lambda_1\mathbf{x})^T \mathbf{y} = (A\mathbf{x})^T \mathbf{y} = \mathbf{x}A^T \mathbf{y} = \mathbf{x}^T A\mathbf{y} = \mathbf{x}^T \lambda_2 \mathbf{y}.$$

The only way this is true is if $\lambda_1 = \lambda_2$ or $\mathbf{x}^T \mathbf{y} = 0$. So, if the eigenvalues are distinct, the eigenvectors are orthogonal.

We have now proven that if all the eigenvalues of a matrix are distinct, then all the eigenvectors are perpendicular. We can scale the eigenvectors so that they all have length 1, and are therefore orthonormal.

So, if A is symmetric, we can diagonalize it as

$$A = Q\Lambda Q^T = \begin{pmatrix} \mathbf{x}_1 & \cdots & \mathbf{x}_n \end{pmatrix} \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} \begin{pmatrix} \mathbf{x}_1^T \\ \vdots \\ \mathbf{x}_n^T \end{pmatrix}$$

Note this can be rewritten as

$$A = \lambda_1 \mathbf{x}_1 \mathbf{x}_1^T + \cdots + \lambda_n \mathbf{x}_n \mathbf{x}_n^T$$

where the $\mathbf{x}_i \mathbf{x}_i^T$ terms are the projection matrices onto the linear spaces spanned by the respective \mathbf{x}_i . These are called the *eigenspaces*.

The book proves at the end of the section that this is true for *any* symmetric matrix, regardless of whether the eigenvalues are repeated or not. So, if A is a symmetric matrix, then A is diagonalizable, and can be written as

$$A = Q\Lambda Q^{-1} = Q\Lambda Q^T.$$

This is known as the *spectral theorem*. Note that, as we mentioned in the last lecture, not all matrices are diagonalizable. But, all symmetric matrices are.

Example - Find the spectral decomposition $A = Q\Lambda Q^T$ of the matrix

$$A = \begin{pmatrix} -2 & 6 \\ 6 & 7 \end{pmatrix}$$

2 Complex Eigenvalues of Real Matrices

If a real matrix is symmetric then all its eigenvalues are real. However, if a real matrix is not symmetric, it's quite possible that there are complex eigenvalues or even complex eigenvectors. However, these eigenvalues and eigenvectors will come in "conjugate pairs"

$$A\mathbf{x} = \lambda\mathbf{x} \qquad A\bar{\mathbf{x}} = \bar{\lambda}\bar{\mathbf{x}}$$

So, each complex eigenvalue has a conjugate twin.