

# Math 2270 - Lecture 31 : Diagonalizing a Matrix

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This lecture covers *section 6.2* of the textbook.

Today we're going to talk about *diagonalizing* a matrix. What we mean by this is that we want to express the matrix as a product of three matrices in the form:

$$A = S\Lambda S^{-1}$$

where  $\Lambda$  is a diagonal matrix. In particular, the diagonal entries of  $\Lambda$  will be the eigenvalues of  $A$ , and the columns of  $S$  will be the corresponding eigenvectors. Having  $A$  in this form can greatly simplify many calculations, particularly calculations involving powers of  $A$ .

The assigned problems for this section are:

Section 6.2 - 1, 2, 15, 16, 26

## 1 The Basic Formula

Suppose the matrix  $A$  has  $n$  distinct eigenvalues. In this case there will be  $n$  corresponding linearly independent eigenvectors.<sup>1</sup> Call these eigenvectors  $\mathbf{x}_1, \dots, \mathbf{x}_n$ . The matrix  $S$  is formed by making these eigenvectors its columns:

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<sup>1</sup>This is proven in the textbook. We'll take it as given.

$$S = \begin{pmatrix} \mathbf{x}_1 & \cdots & \mathbf{x}_n \end{pmatrix}$$

Now, the product  $AS$  will be:

$$A \begin{pmatrix} \mathbf{x}_1 & \cdots & \mathbf{x}_n \end{pmatrix} = \begin{pmatrix} \lambda_1 \mathbf{x}_1 & \cdots & \lambda_n \mathbf{x}_n \end{pmatrix}.$$

If we define  $\Lambda$  to be the diagonal matrix with diagonal entries equal to the eigenvalues of  $A$ , then we also have

$$S\Lambda = \begin{pmatrix} \mathbf{x}_1 & \cdots & \mathbf{x}_n \end{pmatrix} \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} = \begin{pmatrix} \lambda_1 \mathbf{x}_1 & \cdots & \lambda_n \mathbf{x}_n \end{pmatrix}.$$

These two matrices are equal and so we get

$$AS = S\Lambda.$$

Now, all the eigenvectors of  $A$  are linearly independent, and so the matrix  $S$  has full rank, which means it's invertible. Therefore, we have

$$A = S\Lambda S^{-1}.$$

This form,  $S\Lambda S^{-1}$  is called *diagonalizing* the matrix  $A$ .

Example - Diagonalize the matrix

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

First, we calculate the eigenvalues

$$\begin{vmatrix} 1-\lambda & 1 \\ 1 & 0-\lambda \end{vmatrix} = -\lambda(1-\lambda) - 1 = \lambda - \lambda^2 - 1 \\ = \lambda^2 - \lambda - 1$$

We find the roots with the quadratic equation

$$\lambda = \frac{1 \pm \sqrt{(-1)^2 - 4(1)(-1)}}{2} = \frac{1 \pm \sqrt{5}}{2}$$

So, the eigenvalues are  $\lambda_1 = \frac{1+\sqrt{5}}{2}$ ,  $\lambda_2 = \frac{1-\sqrt{5}}{2}$

The eigenvectors are

$$\begin{pmatrix} \frac{1-\sqrt{5}}{2} & 1 \\ 1 & -\frac{1-\sqrt{5}}{2} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \frac{1+\sqrt{5}}{2} \\ 1 \end{pmatrix}, \text{ and } \begin{pmatrix} \frac{1-\sqrt{5}}{2} \\ 1 \end{pmatrix} \text{ or, } \begin{pmatrix} \lambda_1 \\ 1 \end{pmatrix}, \begin{pmatrix} \lambda_2 \\ 1 \end{pmatrix}$$

$$S = \begin{pmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{pmatrix} \quad S^{-1} = \frac{1}{\lambda_1 - \lambda_2} \begin{pmatrix} 1 & -\lambda_2 \\ -1 & \lambda_1 \end{pmatrix}$$

$$3 \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = \frac{1}{\lambda_1 - \lambda_2} \begin{pmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} 1-\lambda_2 \\ -1-\lambda_1 \end{pmatrix}$$

## 2 Diagonalization, Matrix Powers, and Fibonacci

Now, one of the most useful things about the diagonalization of a matrix is that it can make it much, much easier to take powers of the matrix. This is because

$$A^k = (S\Lambda S^{-1})(S\Lambda S^{-1}) \cdots (S\Lambda S^{-1}) = S\Lambda^k S^{-1}.$$

The middle term,  $\Lambda^k$ , is the power of a diagonal matrix, which is comparatively very easy to calculate.

$$\begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}^k = \begin{pmatrix} \lambda_1^k & & \\ & \ddots & \\ & & \lambda_n^k \end{pmatrix}.$$

We'll demonstrate this with an extended example about computing Fibonacci numbers.

The Fibonacci numbers are a very old sequence of numbers defined recursively by the formula

$$F_{k+2} = F_{k+1} + F_k,$$

with a seed of  $F_0 = 0, F_1 = 1$ .<sup>2</sup> From here we can get  $F_2 = 1, F_3 = 2, F_4 = 3, F_5 = 5, \dots$  Now, what we'd like to do is find a closed-form formula for calculating a number  $F_k$  in this sequence. That way, in order to find  $F_k$ , we wouldn't first need to calculate all the Fibonacci numbers up to  $F_{k-1}$ . We can derive such a formula using eigenvalues, eigenvectors, and diagonalization.

We note that if we define a vector whose components are two consecutive terms of the Fibonacci sequence,  $\mathbf{u}_k = \begin{pmatrix} F_{k+1} \\ F_k \end{pmatrix}$ , then the action of the matrix  $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$  on the vector  $\mathbf{u}_k$  produces the vector  $\mathbf{u}_{k+1}$ .

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<sup>2</sup>You may remember the Fibonacci numbers from "The Da Vinci Code", if from nowhere else.

$$A\mathbf{u}_k = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} F_{k+1} \\ F_k \end{pmatrix} = \begin{pmatrix} F_{k+1} + F_k \\ F_{k+1} \end{pmatrix} = \begin{pmatrix} F_{k+2} \\ F_{k+1} \end{pmatrix} = \mathbf{u}_{k+1}$$

Now, our initial "seed" vector will be

$$\mathbf{u}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

If we take the eigenvalues and eigenvectors we calculated for  $A$  in the example above<sup>3</sup> we can express  $\mathbf{u}_0$  as

$$\mathbf{u}_0 = \frac{\mathbf{x}_1 - \mathbf{x}_2}{\lambda_1 - \lambda_2}.$$

The successive vectors in the sequence will be:

$$\mathbf{u}_k = A^k \mathbf{u}_0 = \frac{(\lambda_1)^k \mathbf{x}_1 - \lambda_2^k \mathbf{x}_2}{\lambda_1 - \lambda_2}.$$

In this formula we use the fact that  $A^k \mathbf{x}_1 = \lambda_1^k \mathbf{x}_1$  and  $A^k \mathbf{x}_2 = \lambda_2^k \mathbf{x}_2$ .

*Example* - Calculate  $F_{18}$ , the 18th<sup>4</sup> Fibonacci number.

$$F_{18} = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^{18} - \left(\frac{1-\sqrt{5}}{2}\right)^{18}}{\sqrt{5}} = 2,584$$

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<sup>3</sup>OK, so if you're reading these lecture notes before hand, you might be able to figure out the answer to the example problem without actually having to do it. So sue me.

<sup>4</sup>Or 19, depending on if you start your counting at 0 or 1.

### 3 Not All Matrices Are Diagonalizable

We've assumed up to this point that all eigenvalues are distinct. If this is the case then there will be  $n$  corresponding eigenvectors, and they will all be linearly independent. This is proven in the textbook. The proof is pretty straightforward, but I'm going to skip it here and ask you to read the textbook if you're interested in it. The important thing for us is the result.

Now, of course, it's not always the case that all the eigenvalues of a matrix are distinct, and in these cases it will not always be true that we can find  $n$  linearly independent eigenvectors. If we can't find  $n$  linearly independent eigenvectors, then we can't diagonalize the matrix. So, not all matrices are diagonalizable.

For example, the matrix

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

has characteristic equation  $\lambda^2 = 0$ , and therefore has only one eigenvalue,  $\lambda = 0$ , of order 2. The only eigenvectors are the non-zero constant multiples of

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

We cannot form our matrix  $S$  from just this one eigenvector, and therefore we cannot diagonalize  $A$ . So, whenever a matrix  $A$  has *distinct* eigenvalues it will be diagonalizable, but if it has repeated eigenvalues it might not be.