Math 2270 - Lecture 2: Lengths and Dot Products

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1 The Dot Product

Last time we learned how to add vectors together (the result is another vector), how to multiply a vector by a scalar (again, we get a vector), and how to combine these two operations to form linear combinations.

There's no obvious geometric way of multiplying vectors (I mean, what is 10 mph North times 5 mph West...), but we do have a notion of an "inner product" or "dot product" of two vectors.

Definition The *dot product* or *inner product* of two vectors (u_1, u_2) and (v_1, v_2) is the number:

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2.$$

For two n-dimensional vectors, the above definition generalizes as:

$$\mathbf{u} \cdot \mathbf{v} = \sum_{i=1}^{n} u_i v_i.$$

Note that the order of the dot product doesn't matter. So, $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$.

Example - Calculate the dot product $\mathbf{u} \cdot \mathbf{v}$ for the vectors

$$\mathbf{u} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \mathbf{v} = \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix}.$$

$$1 \times 2 + 2 \times 3 + 3 \times 0 = \boxed{8}$$

2 Lengths and Unit Vectors

If we take the dot product of a nonzero vector with itself, we get a number that is always positive.

Example - What is the dot product $\mathbf{v} \cdot \mathbf{v}$ where $\mathbf{v} = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$?

$$1^2 + 3^2 + 2^2 = \boxed{14}$$

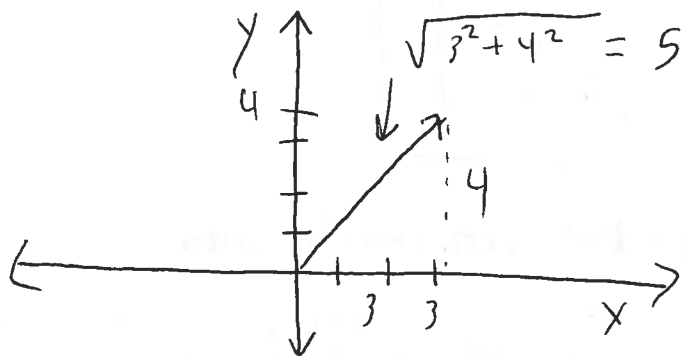
We define the *length* of a vector to be the square root of the dot product of that vector with itself.

Example - What is the length of the vector $\mathbf{v}$ from the example above?

$$\sqrt{\vec{v}\cdot\vec{v}} = \boxed{\sqrt{14}}$$

The length of a vector $\mathbf{v}$ is usually written $||\mathbf{v}||$. So, $||\mathbf{v}|| = \sqrt{\mathbf{v}\cdot\mathbf{v}}$.

Now, this definition makes geometric sense. Suppose, for example, we have the vector $\begin{pmatrix} 3 & 4 \end{pmatrix}$. If we draw this vector in the xy-plane, with its tip at the origin, we can apply the Pythagorean theorem to see that its length is 5.



In general, for a vector with components $\begin{pmatrix} x & y \end{pmatrix}$ the Pythagorean theorem tells us its length is $\sqrt{x^2+y^2}$, and our definition of length in terms of the dot product is just a generalization of this idea.

A *unit* vector is a vector of length 1. Some unit vectors are

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}.$$

If we have a vector $\mathbf{v}$, a unit vector in the same direction as $\mathbf{v}$ is usually written $\hat{\mathbf{v}}$, and is defined as:

$$\hat{\mathbf{v}} = \frac{\mathbf{v}}{||\mathbf{v}||}.$$

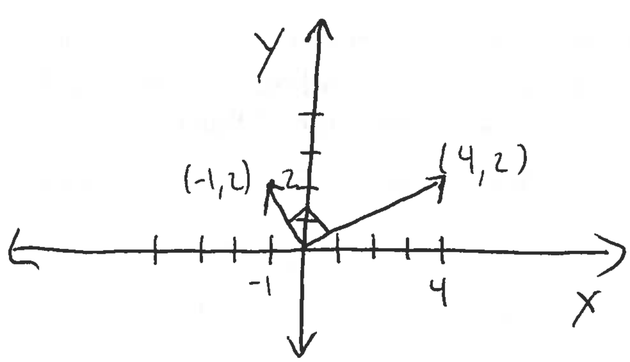
Example What are the components of a unit vector in the same direction as the vector $\mathbf{v} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$?

$$||\vec{v}|| = \sqrt{14}$$

$$\Rightarrow \quad \hat{v} = \boxed{\begin{pmatrix} \frac{1}{\sqrt{14}} \\ \frac{2}{\sqrt{14}} \\ \frac{3}{\sqrt{14}} \end{pmatrix}}$$

3 The Angle Between Two Vectors

The vectors $\begin{pmatrix} 4 & 2 \end{pmatrix}$ and $\begin{pmatrix} -1 & 2 \end{pmatrix}$ have dot product $-4 + 4 = 0$. If we draw these two vectors, we see they're perpendicular.



This isn't a coincidence. Two vectors are perpendicular if and only if their dot product is 0.

In general, the dot product can be used to measure the angle between any two vectors. If θ is the angle between vectors $\mathbf{u}$ and $\mathbf{v}$, then the dot product of $\mathbf{u}$ and $\mathbf{v}$ is related to the angle between them by the formula:

$$\mathbf{u} \cdot \mathbf{v} = ||\mathbf{u}||||\mathbf{v}||cos\theta.$$

If the dot product is negative, then $\theta > 90°$, while if the dot product is positive, then $\theta < 90^0$. If the dot product is 0, then $\theta = 90°$ on the nose.

Example - Find the angle θ between the vectors

$$\mathbf{u} = \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix} \text{ and } \mathbf{v} = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}.$$

$\vec{u} \cdot \vec{v} = 0 \Rightarrow \cos\theta = 0 \Rightarrow \boxed{\theta = 90^\circ}$

We can use our angle relation to derive two of the most famous inequalities in mathematics.

Schwarz Inequality $|\mathbf{u} \cdot \mathbf{v}| \leq ||\mathbf{u}||||\mathbf{v}||$.

Triangle Inequality $||\mathbf{u} + \mathbf{v}|| \leq ||\mathbf{u}|| + ||\mathbf{v}||$.

Example - Derive the triangle inequality.

$$(\vec{u}+\vec{v})\cdot(\vec{u}+\vec{v}) = \vec{u}\cdot\vec{u} + 2\,\vec{u}\cdot\vec{v} + \vec{v}\cdot\vec{v}$$

$$(\|\vec{u}\| + \|\vec{v}\|)^2 = \vec{u}\cdot\vec{u} + 2\|\vec{u}\|\,\|\vec{v}\| + \vec{v}\cdot\vec{v}$$

Now, $\vec{u}\cdot\vec{v} \le \|\vec{u}\|\,\|\vec{v}\|$, so

$$(\vec{u}+\vec{v})\cdot(\vec{v}+\vec{v}) = \|\vec{u}+\vec{v}\|^2 \le (\|\vec{u}\| + \|\vec{v}\|^2$$

$$\Rightarrow \boxed{\|\vec{u}+\vec{v}\| \le \|\vec{u}\| + \|\vec{v}\|}$$

I leave the Schwarz inequality as an exercise for you to do on your own. Try it!