Math 2270 - Lecture 29 : Determinants and Volumes

Dylan Zwick

Fall 2012

This lecture covers the second part of *section 5.3* from the textbook.

Today we're going to learn about a practical geometric application of the determinant. Its absolute value tells you the volume of an n-dimensional hyperbox! In terms with which you may be more familiar, we can use the determinant to calculate the volume of a parallelogram in 2 dimensions, and a parallelepipid (OK, we're already getting into fancy terms) in 3 dimensions.

When we finally start talking about linear transformations, we'll return to this to find a geometrically obvious, intuitive reason why a matrix is invertible if and only if its determinant is non-zero.

The assigned problems for this section are:

Section 5.3 - 1, 6, 7, 8, 16

1 How to Use Determinants to Calculate the Area of a Triangle or a Parallelogram

Three points in the plane, $(x_1, y_1), (x_2, y_2), (x_3, y_3)$ determine a triangle.
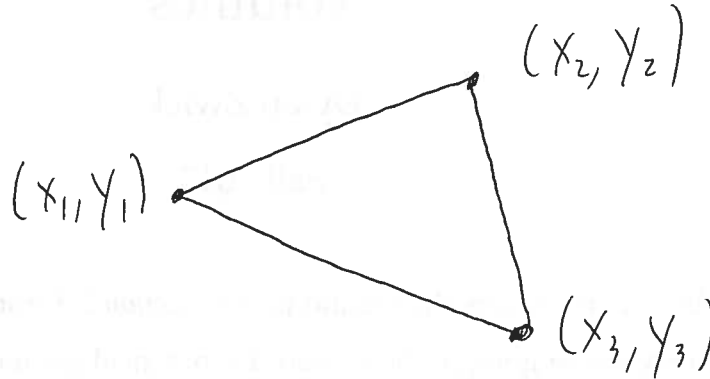
(x_2, y_2)

(x_1, y_1)

(x_3, y_3)

Wouldn't it be nice if there were an easy to remember formula for calculating the area of this triangle based on the three points? Well, you're in luck, because there is! We just use the determinant.
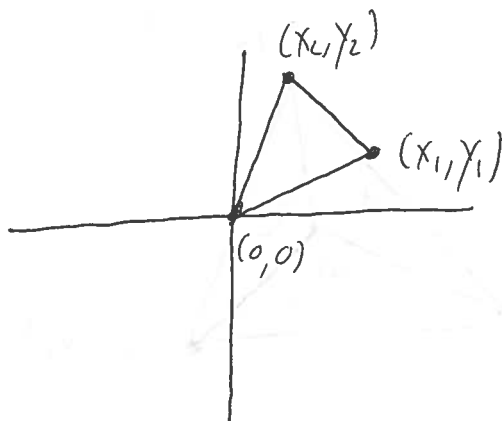
$$\text{Area of Triangle} = abs\left(\frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} \right)$$

So, the area is the absolute value of $\frac{1}{2}$ the determinant of the matrix:

$$\begin{pmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{pmatrix}.$$

If one of the points, say the third, is the origin (so $(x_3, y_3) = (0, 0)$), then the area formula simplifies to a 2×2 determinant:

$$abs\left(\frac{1}{2}\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}\right)$$
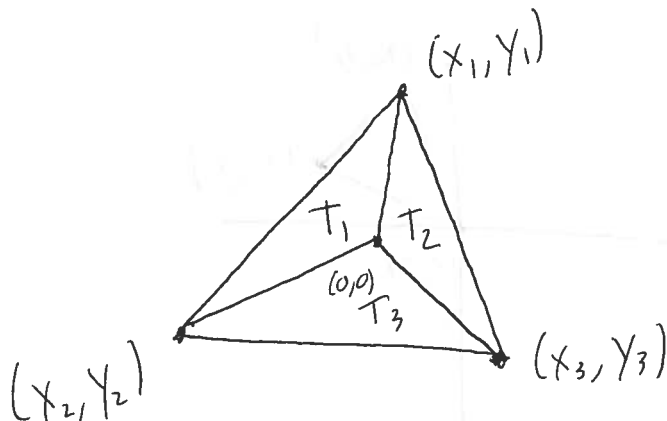

Example - Prove this special 2×2 formula using our formula for three points.

If $(x_3, y_3) = (0,0)$ then

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}$$

just by doing a cofactor expansion along row 3.

3

We can go the other way and use this 2×2 formula to derive the 3×3 formula. First, assume that the origin is is enclosed by the triangle


In this case we can subdivide the triangle into three parts, as shown above, and add up the areas for these three parts using the 2×2 formula to get:

$$Area = Area(T_1) + Area(T_2) + Area(T_3)$$

$$= abs\left(\frac{1}{2}\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}\right) + abs\left(\frac{1}{2}\begin{vmatrix} x_1 & y_1 \\ x_3 & y_3 \end{vmatrix}\right) + abs\left(\frac{1}{2}\begin{vmatrix} x_2 & y_2 \\ x_3 & y_3 \end{vmatrix}\right)$$

$$= \frac{1}{2}(x_1 y_2 - x_2 y_1) + \frac{1}{2}(x_3 y_1 - x_1 y_3) + \frac{1}{2}(x_2 y_3 - x_3 y_2) = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

We've played a little fast and loose with absolute values here. Sometimes the signs won't work out quite the way I wrote them above, but keep calm, the formula still works, and proving it just requires a little bookkeeping. We won't worry about it.

4

Note that even if the origin isn't enclosed by the triangle
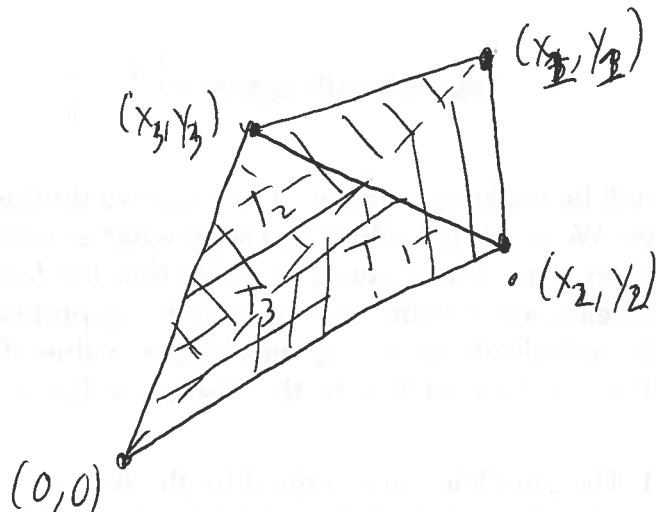
the formula still works. It just calculates the areas of T_1 and T_2, and then subtracts the area of T_3.

OK, so let's see if we can derive the formula for the area of a triangle when one of the points is the origin. We can double the triangle to get a parallelogram
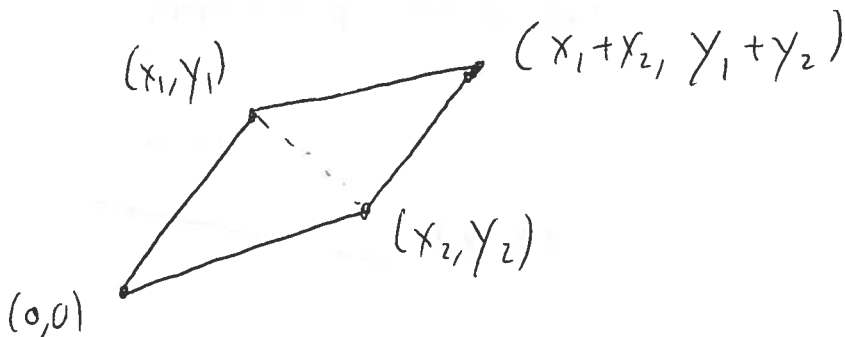
and the area of the triangle with be $\frac{1}{2}$ the area of the parallelogram. So,
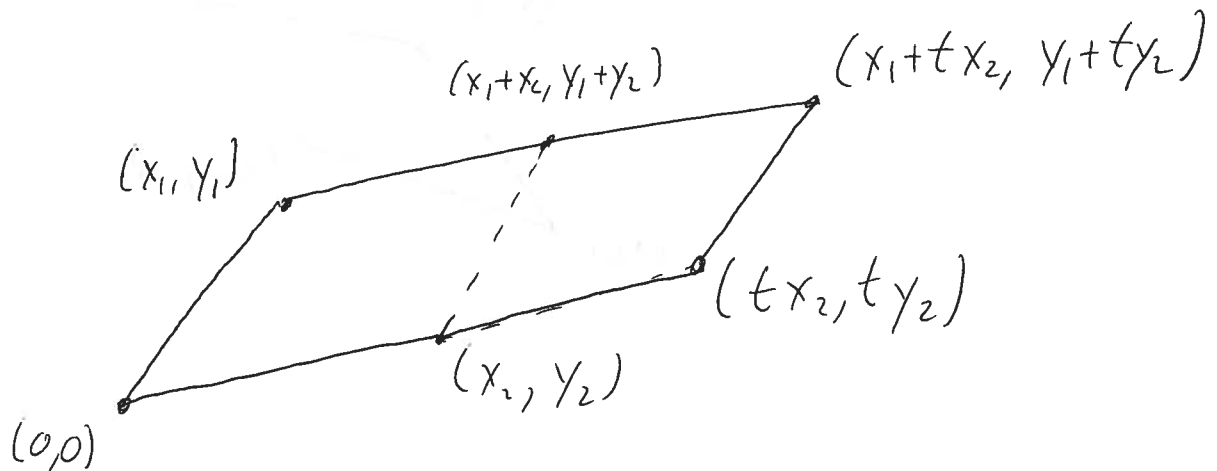
if we can prove

$$\text{Area of Parallelogram} = \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}$$

then we'll be cooking with gas. How can we do this? Well, there are many ways. We're going to do it in a somewhat esoteric way, but there's a good reason why. We're going to prove that the formula for the area of the parallelogram satisfies the three defining properties of the determinant. Or, technically speaking, the absolute value of the determinant. Then, we'll know the area *must* be the absolute value of the determinant.
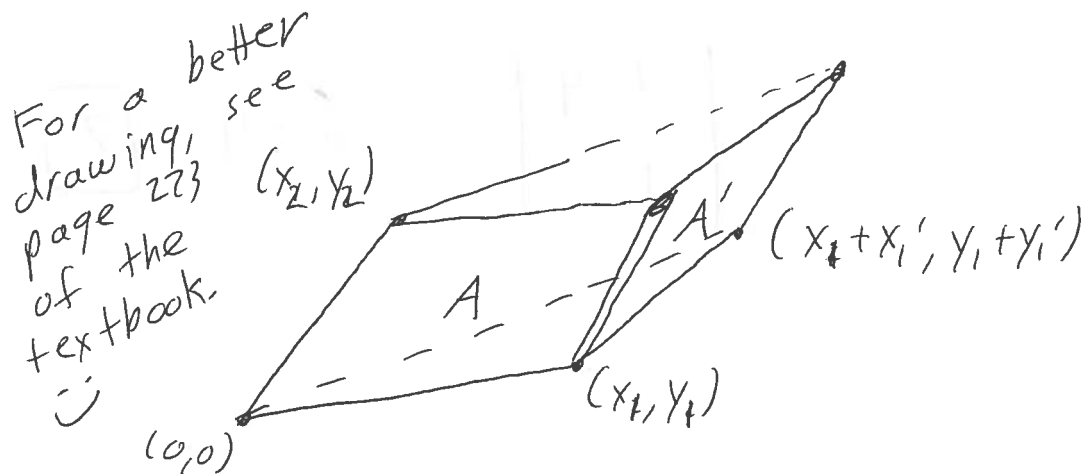
Property 1 The parallelogram defined by the two unit vectors $(1, 0)$ and $(0, 1)$ is just the unit square, which has area 1. So, $det(I) = 1$.

Property 2 If we switch two rows the parallelogram remains unchanged (we just change how we label the points). Switching the rows changes the sign of the determinant, but not the absolute value. So, we're still good.

Property 3 If we multiply one of the points by t, the area goes from A to tA, as shown in the picture below:

6

If we add (x_1', y_1') to the point (x_1, y_1) the new area if the sum of the areas T and T' below:

For a better drawing, see page 273 of the textbook. ☺

(x_2, y_2)

A'

$(x_1 + x_1', y_1 + y_1')$

A

(x_1, y_1)

$(0,0)$

where we note that the area we miss in the triangle on top we make up for in the triangle at the bottom. The areas of T and T' are the areas we'd get for the points (x_1, y_1) and (x_2, y_2), and (x_1', y_1') and (x_2, y_2), respectively. So, property 3 is satisfied.

The three defining properties are satisfied, so the area of the parallelogram *must* be the absolute value of the determinant.

Now, why did we do it this way? Because the same reasoning used above works in an arbitraty number of dimensions. If we have an n-dimensional box defined by n vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$, then the n-dimensional volume of that box is just the absolute value of the determinant of the matrix whose columns are the given vectors. We'll revisit this again when we talk about linear transformations.

Example - Calculate the area of the parallelogram defined by the points $(3, 4)$ and $(1, 2)$:

$$\begin{vmatrix} 3 & 4 \\ 1 & 2 \end{vmatrix} = 6 - 4 = \boxed{2}$$