Math 2270 - Lecture 28 : Cramer's Rule

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This lecture covers the first part of *section 5.3* from the textbook.

We already know, and in fact have known for quite a while, how to solve $A\mathbf{x} = \mathbf{b}$. Namely, we use elimination. Cramer's rule gives us another way to do it. I think it's fair to say that Cramer's rule works better in theory than it works in practice. What I mean by this is that, in practice, when you want to actually solve a system of equations you'd never use Cramer's rule. It's not nearly as efficient as elimination. However, when you need to actually prove things about matrices and linear transformations, Cramer's rule can be very, very useful.

The assigned problems for this section are:

Section 5.3 - 1, 6, 7, 8, 16

1 Cramer's Rule

Cramer's rule begins with the clever observation

$$\begin{vmatrix} x_1 & 0 & 0 \\ x_2 & 1 & 0 \\ x_3 & 0 & 1 \end{vmatrix} = x_1$$

That is to say, if you replace the first column of the identity matrix with the vector $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ the determinant is x_1. Now, we've illustrated this for the 3×3 case and for column one, but there's nothing special about a 3×3 identity matrix or the first column. In general, if you replace the ith column of an $n \times n$ identity matrix with a vector $\mathbf{x}$, the determinant of the matrix you get will be x_i, the ith component of $\mathbf{x}$.

Well, that's great. Now what do we do with this information? Well, note that if $A\mathbf{x} = \mathbf{b}$ then

$$\left(\quad A \quad \right) \begin{pmatrix} x_1 & 0 & 0 \\ x_2 & 1 & 0 \\ x_3 & 0 & 1 \end{pmatrix} = \begin{pmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{pmatrix}.$$

If we take determinants of both sides, and note the determinant is multiplicative, we get

$$det(A)x_1 = det(B_1)$$

where B_1 is the matrix we get when we replace column 1 of A by the vector $\mathbf{b}$. So,

$$x_1 = \frac{det(B_1)}{det(A)}.$$

Now, again, there's nothing special here about column 1, or about them being 3×3 matrices. In general if we have the relation $A\mathbf{x} = \mathbf{b}$ then the ith component of $\mathbf{x}$ will be

$$x_i = \frac{det(B_i)}{det(A)},$$

where B_i is the matrix we get by replacing column i of A with $\mathbf{b}$.

Example
Use Cramer's rule to solve for the vector **x**:

$$\begin{pmatrix} -1 & 2 & -3 \\ 2 & 0 & 1 \\ 3 & -4 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$$

$$\det(A) = -2 \begin{vmatrix} 2 & -3 \\ -4 & 4 \end{vmatrix} - 1 \begin{vmatrix} -1 & 2 \\ 3 & -4 \end{vmatrix}$$

$$= -2(-4) - 1(-2) = 10$$

$$\det(B_1) = \begin{vmatrix} 1 & 2 & -3 \\ 0 & 0 & 1 \\ 2 & -4 & 4 \end{vmatrix} = - \begin{vmatrix} 1 & 2 \\ 2 & -4 \end{vmatrix} = 8$$

$$\det(B_2) = \begin{vmatrix} -1 & 1 & -3 \\ 2 & 0 & 1 \\ 3 & 2 & 4 \end{vmatrix} = -2 \begin{vmatrix} 1 & -3 \\ 2 & 4 \end{vmatrix} - \begin{vmatrix} -1 & 1 \\ 3 & 2 \end{vmatrix}$$

$$= -2(10) - (-5) = -15$$

$$\det(B_3) = \begin{vmatrix} -1 & 2 & 1 \\ 2 & 0 & 0 \\ 3 & -4 & 2 \end{vmatrix} = -2 \begin{vmatrix} 2 & 1 \\ -4 & 2 \end{vmatrix} = -16$$

$$x_1 = \frac{8}{10} = \frac{4}{5}, \quad x_2 = \frac{-15}{10} = -\frac{3}{2} \quad x_3 = -\frac{16}{10} = -\frac{8}{5}$$

$$\vec{x} = \begin{pmatrix} \frac{4}{5} \\ -\frac{3}{2} \\ -\frac{8}{5} \end{pmatrix}$$

2 Calculating Inverses using Cramer's Rule

When we multiply two matrices together, the product (in the 3×3 case) is

$$AB = \begin{pmatrix} & A & \end{pmatrix} \begin{pmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \mathbf{b}_3 \end{pmatrix} = \begin{pmatrix} A\mathbf{b}_1 & A\mathbf{b}_2 & A\mathbf{b}_3 \end{pmatrix}.$$

Here we've used two square, 3×3 matrices, but the idea works generally. Each column of the product is the left matrix A multiplied by the appropriate column of the right matrix B.

To find the inverse of a matrix A, we want to find another matrix B such that $AB = I$. Writing it out for the 3×3 case we have

$$\begin{pmatrix} & A & \end{pmatrix} \begin{pmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \mathbf{b}_3 \end{pmatrix} = \begin{pmatrix} A\mathbf{b}_1 & A\mathbf{b}_2 & A\mathbf{b}_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

If this is the case then $B = A^{-1}$. So, finding the inverse of an $n \times n$ square matrix A amounts to solving n equations of the form

$$A\mathbf{x}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, A\mathbf{x}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \ldots, A\mathbf{x}_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}.$$

The columns of A^{-1} will be the vectors $\mathbf{x}_1, \ldots, \mathbf{x}_n$. Well, we can find these vectors using Cramer's rule.

Going back to the 3×3 case, suppose we want to find $(A^{-1})_{32}$. This will be the third component of the vector $\mathbf{x}_2$. Cramer's rule tells us this will be:

$$(A^{-1})_{32} = (\mathbf{x}_2)_3 = \frac{1}{det(A)} \begin{vmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 1 \\ a_{31} & a_{32} & 0 \end{vmatrix}.$$

We can calculate the determinant

$$\begin{vmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 1 \\ a_{31} & a_{32} & 0 \end{vmatrix}$$

by a cofactor expansion along colum 3 to get

$$\begin{vmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 1 \\ a_{31} & a_{32} & 0 \end{vmatrix} = (-1)^{2+3} \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} = (-1)^{2+3} det(M_{23}) = C_{23}$$

where M_{23} is the submatrix we get by eliminating column 3 and row 2 of A, and C_{23} is the $(2,3)$ cofactor of A. So, what we have is

$$(A^{-1})_{32} = \frac{C_{23}}{det(A)}.$$

Note that the index $(3,2)$ of the inverse is *switched* for the index $(2,3)$ of the cofactor.

This method works generally. The component $(A^{-1})_{ij}$ will be

$$(A_{ij})^{-1} =$$

$$\frac{1}{det(A)} \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1(i-1)} & 0 & a_{1(i+1)} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2(i-1)} & 0 & a_{2(i+1)} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{(j-1)1} & a_{(j-1)2} & \cdots & a_{(j-1)(i-1)} & 0 & a_{(j-1)(i+1)} & \cdots & a_{(i-1)n} \\ a_{j1} & a_{j2} & \cdots & a_{j(i-1)} & 1 & a_{j(i+1)} & \cdots & a_{jn} \\ a_{(j+1)1} & a_{(j+1)2} & \cdots & a_{(j+1)(i-1)} & 0 & a_{(j+1)(i+1)} & \cdots & a_{(j+1)n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{n(i-1)} & 0 & a_{n(i+1)} & \cdots & a_{nn} \end{vmatrix}.$$

The big, narsty determinant on the right is the determinant of the matrix that we get when we replace column i by the jth column of the identity, a.k.a. the unit vector in the jth direction. Now, if we do a cofactor expansion along column i of that matrix, we can see the determinant is equal to the cofactor C_{ji} of A. Writing this result succinctly, we have

$$(A^{-1})_{ij} = \frac{C_{ji}}{det(A)}.$$

This formula works in general for any $n \times n$ matrix A. Please remember, again, that this isn't how you'd calculate the inverse in practice. You'd use elimination. But for theoretical work this formula can be very useful.

We end with some notation. For the matrix A we can define a matrix of cofactors

$$\begin{pmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{pmatrix}.$$

The transpose of this matrix

$$\begin{pmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{pmatrix}$$

is called the *adjoint* of the matrix A, or $adj(A)$. Using this terminology we can write

$$A^{-1} = \left(\frac{1}{det(A)}\right) adj(A).$$

We can use this to derive general formulas for an $n \times n$ determinant in the same way we derived the formula

$$A^{-1} = \frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

for a 2×2 matrix. However, as n gets larger, the formulas get messier and messier.

Example - Calculate the determinant and the adjoint of the matrix A, and use them to find the inverse of A.

$$A = \begin{pmatrix} -1 & 3 & 2 \\ 0 & -2 & 1 \\ 1 & 0 & 2 \end{pmatrix}.$$

$$C_{11} = \begin{vmatrix} -2 & 1 \\ 0 & 2 \end{vmatrix} = -4 \qquad C_{12} = -\begin{vmatrix} 0 & 1 \\ 1 & 2 \end{vmatrix} = +1 \qquad C_{13} = \begin{vmatrix} 0 & -2 \\ 1 & 0 \end{vmatrix} = 2$$

$$C_{21} = -\begin{vmatrix} 3 & 2 \\ 0 & 2 \end{vmatrix} = -6 \qquad C_{22} = \begin{vmatrix} -1 & 2 \\ 1 & 2 \end{vmatrix} = -4 \qquad C_{23} = -\begin{vmatrix} -1 & 3 \\ 1 & 0 \end{vmatrix} = 3$$

$$C_{31} = \begin{vmatrix} 3 & 2 \\ -2 & 1 \end{vmatrix} = 7 \qquad C_{32} = -\begin{vmatrix} -1 & 2 \\ 0 & 1 \end{vmatrix} = 1 \qquad C_{33} = \begin{vmatrix} -1 & 3 \\ 0 & -2 \end{vmatrix} = 2$$

$$C = \begin{pmatrix} -4 & 1 & 2 \\ -6 & -4 & 3 \\ 7 & 1 & 2 \end{pmatrix} \qquad \mathrm{adj}(A) = C^T = \begin{pmatrix} -4 & -6 & 7 \\ 1 & -4 & 1 \\ 2 & 3 & 2 \end{pmatrix}$$

$$\det(A) = 0 \cdot C_{21} + (-2)C_{22} + 1C_{23}$$

$$= 0 + 8 + 3 = 11$$

$$A^{-1} = \frac{1}{\det(A)} \mathrm{adj}(A) = \frac{1}{11} \begin{pmatrix} -4 & -6 & 7 \\ 1 & -4 & 1 \\ 2 & 3 & 2 \end{pmatrix}$$