

# Math 2270 - Lecture 28 : Cramer's Rule

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This lecture covers the first part of *section 5.3* from the textbook.

We already know, and in fact have known for quite a while, how to solve  $Ax = \mathbf{b}$ . Namely, we use elimination. Cramer's rule gives us another way to do it. I think it's fair to say that Cramer's rule works better in theory than it works in practice. What I mean by this is that, in practice, when you want to actually solve a system of equations you'd never use Cramer's rule. It's not nearly as efficient as elimination. However, when you need to actually prove things about matrices and linear transformations, Cramer's rule can be very, very useful.

The assigned problems for this section are:

Section 5.3 - 1, 6, 7, 8, 16

## 1 Cramer's Rule

Cramer's rule begins with the clever observation

$$\begin{vmatrix} x_1 & 0 & 0 \\ x_2 & 1 & 0 \\ x_3 & 0 & 1 \end{vmatrix} = x_1$$

That is to say, if you replace the first column of the identity matrix with the vector  $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$  the determinant is  $x_1$ . Now, we've illustrated this

for the  $3 \times 3$  case and for column one, but there's nothing special about a  $3 \times 3$  identity matrix or the first column. In general, if you replace the  $i$ th column of an  $n \times n$  identity matrix with a vector  $\mathbf{x}$ , the determinant of the matrix you get will be  $x_i$ , the  $i$ th component of  $\mathbf{x}$ .

Well, that's great. Now what do we do with this information? Well, note that if  $A\mathbf{x} = \mathbf{b}$  then

$$\begin{pmatrix} & & \\ & A & \\ & & \end{pmatrix} \begin{pmatrix} x_1 & 0 & 0 \\ x_2 & 1 & 0 \\ x_3 & 0 & 1 \end{pmatrix} = \begin{pmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{pmatrix}.$$

If we take determinants of both sides, and note the determinant is multiplicative, we get

$$\det(A)x_1 = \det(B_1)$$

where  $B_1$  is the matrix we get when we replace column 1 of  $A$  by the vector  $\mathbf{b}$ . So,

$$x_1 = \frac{\det(B_1)}{\det(A)}.$$

Now, again, there's nothing special here about column 1, or about them being  $3 \times 3$  matrices. In general if we have the relation  $A\mathbf{x} = \mathbf{b}$  then the  $i$ th component of  $\mathbf{x}$  will be

$$x_i = \frac{\det(B_i)}{\det(A)},$$

where  $B_i$  is the matrix we get by replacing column  $i$  of  $A$  with  $\mathbf{b}$ .

*Example*

Use Cramer's rule to solve for the vector  $\mathbf{x}$ :

$$\begin{pmatrix} -1 & 2 & -3 \\ 2 & 0 & 1 \\ 3 & -4 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$$

## 2 Calculating Inverses using Cramer's Rule

When we multiply two matrices together, the product (in the  $3 \times 3$  case) is

$$AB = \begin{pmatrix} & A & \end{pmatrix} \begin{pmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \mathbf{b}_3 \end{pmatrix} = \begin{pmatrix} A\mathbf{b}_1 & A\mathbf{b}_2 & A\mathbf{b}_3 \end{pmatrix}.$$

Here we've used two square,  $3 \times 3$  matrices, but the idea works generally. Each column of the product is the left matrix  $A$  multiplied by the appropriate column of the right matrix  $B$ .

To find the inverse of a matrix  $A$ , we want to find another matrix  $B$  such that  $AB = I$ . Writing it out for the  $3 \times 3$  case we have

$$\begin{pmatrix} & A & \end{pmatrix} \begin{pmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \mathbf{b}_3 \end{pmatrix} = \begin{pmatrix} A\mathbf{b}_1 & A\mathbf{b}_2 & A\mathbf{b}_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

If this is the case then  $B = A^{-1}$ . So, finding the inverse of an  $n \times n$  square matrix  $A$  amounts to solving  $n$  equations of the form

$$A\mathbf{x}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, A\mathbf{x}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \dots, A\mathbf{x}_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}.$$

The columns of  $A^{-1}$  will be the vectors  $\mathbf{x}_1, \dots, \mathbf{x}_n$ . Well, we can find these vectors using Cramer's rule.

Going back to the  $3 \times 3$  case, suppose we want to find  $(A^{-1})_{32}$ . This will be the third component of the vector  $\mathbf{x}_2$ . Cramer's rule tells us this will be:

$$(A^{-1})_{32} = (\mathbf{x}_2)_3 = \frac{1}{\det(A)} \begin{vmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 1 \\ a_{31} & a_{32} & 0 \end{vmatrix}.$$

We can calculate the determinant

$$\begin{vmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 1 \\ a_{31} & a_{32} & 0 \end{vmatrix}$$

by a cofactor expansion along column 3 to get

$$\begin{vmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 1 \\ a_{31} & a_{32} & 0 \end{vmatrix} = (-1)^{2+3} \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} = (-1)^{2+3} \det(M_{23}) = C_{23}$$

where  $M_{23}$  is the submatrix we get by eliminating column 3 and row 2 of  $A$ , and  $C_{23}$  is the  $(2, 3)$  cofactor of  $A$ . So, what we have is

$$(A^{-1})_{32} = \frac{C_{23}}{\det(A)}.$$

Note that the index  $(3, 2)$  of the inverse is *switched* for the index  $(2, 3)$  of the cofactor.

This method works generally. The component  $(A^{-1})_{ij}$  will be

$$(A_{ij})^{-1} = \frac{1}{\det(A)} \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1(i-1)} & 0 & a_{1(i+1)} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2(i-1)} & 0 & a_{2(i+1)} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{(j-1)1} & a_{(j-1)2} & \cdots & a_{(j-1)(i-1)} & 0 & a_{(j-1)(i+1)} & \cdots & a_{(j-1)n} \\ a_{j1} & a_{j2} & \cdots & a_{j(i-1)} & 1 & a_{j(i+1)} & \cdots & a_{jn} \\ a_{(j+1)1} & a_{(j+1)2} & \cdots & a_{(j+1)(i-1)} & 0 & a_{(j+1)(i+1)} & \cdots & a_{(j+1)n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{n(i-1)} & 0 & a_{n(i+1)} & \cdots & a_{nn} \end{vmatrix}.$$

The big, nasty determinant on the right is the determinant of the matrix that we get when we replace column  $i$  by the  $j$ th column of the identity, a.k.a. the unit vector in the  $j$ th direction. Now, if we do a cofactor

expansion along column  $i$  of that matrix, we can see the determinant is equal to the cofactor  $C_{ji}$  of  $A$ . Writing this result succinctly, we have

$$(A^{-1})_{ij} = \frac{C_{ji}}{\det(A)}.$$

This formula works in general for any  $n \times n$  matrix  $A$ . Please remember, again, that this isn't how you'd calculate the inverse in practice. You'd use elimination. But for theoretical work this formula can be very useful.

We end with some notation. For the matrix  $A$  we can define a matrix of cofactors

$$\begin{pmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{pmatrix}.$$

The transpose of this matrix

$$\begin{pmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{pmatrix}$$

is called the *adjoint* of the matrix  $A$ , or  $\text{adj}(A)$ . Using this terminology we can write

$$A^{-1} = \left( \frac{1}{\det(A)} \right) \text{adj}(A).$$

We can use this to derive general formulas for an  $n \times n$  determinant in the same way we derived the formula

$$A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

for a  $2 \times 2$  matrix. However, as  $n$  gets larger, the formulas get messier and messier.

*Example* - Calculate the determinant and the adjoint of the matrix  $A$ , and use them to find the inverse of  $A$ .

$$A = \begin{pmatrix} -1 & 3 & 2 \\ 0 & -2 & 1 \\ 1 & 0 & 2 \end{pmatrix}.$$