Math 2270 - Lecture 27 : Calculating Determinants

Dylan Zwick

Fall 2012

This lecture covers *section 5.2* from the textbook.

In the last lecture we stated and discovered a number of properties about determinants. However, we didn't talk much about how to calculate them. In fact, the only general formula was the nasty formula mentioned at the beginning, namely

$$det(A) = \sum_{\sigma \in S_n} sgn(\sigma) \prod_{i=1}^{n} a_{i\sigma(i)}.$$

We also learned a formula for calculating the determinant in a very special case. Namely, if we have a triangular matrix, the determinant is just the product of the diagonals.

Today, we're going to discuss how that special triangular case can be used to calculate determinants in a very efficient manner, and we'll derive the nasty formula. We'll also go over one other way of calculating a determinnat, the "cofactor expansion", that, to be honest, is not all that useful computationally, but can be very useful when you need to prove things.

The assigned problems for this section are:

Section 5.2 - 1, 3, 11, 15, 16

1 Calculating the Determinant from the Pivots

In practice, the easiest way to calculate the determinant of a general matrix is to use elimination to get an upper-triangular matrix with the same determinant, and then just calculate the determinant of the upper-triangular matrix by taking the product of the diagonal terms, a.k.a. the pivots.

If there are no row exchanges required for the LU decomposition of A, then $A = LU$, and $det(A) = det(L)det(U)$. Both L and U are triangular, and all the terms on the diagonal of L are 1, so $det(L) = 1$. The terms on the diagonal of U, $d_1, d_2, \ldots, d_n$, are the pivots, and $det(U) = d_1 d_2 \cdots d_n$. So, $det(A) = d_1 \cdots d_n$. If a permutation is necessary then we have $PA = LU$, and $det(P) = \pm 1$. So, in general, $det(A) = \pm d_1 \cdots d_n$ where the sign is determined by the permutation matrix P.

Example - Use elimination to calculate the determinant of the matrix

$$A = \begin{pmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{pmatrix} \xrightarrow[\text{Add row 1 to row 3}]{\text{subtract 2x row 1 from row 2}} \begin{pmatrix} 2 & 4 & -2 \\ 0 & 1 & 1 \\ 0 & 1 & 5 \end{pmatrix}$$

Subtract row 2 from row3 →

$$\begin{pmatrix} 2 & 4 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 4 \end{pmatrix} \Rightarrow det(A) = (2)(1)(4) = \boxed{8}$$

Now, if no permutation is involved, so $A = LU$, then the first k pivots are completely determined by the upper left $k \times k$ submatrix of the matrix A. If we denote the upper left $k \times k$ submatrix of A by A_k then we have

$$det(A_k) = d_1 \cdots d_k$$

and the kth pivot is

$$d_k = \frac{det(A_k)}{det(A_{k-1})}.$$

So, we can use pivots to calculate determinants, but we can also (assuming we don't need row exchanges) use determinants to calculate pivots!

2 The Big Honkin' Formula

Alright, it's been coming for a while now. The formal definition of the determinant is

$$det(A) = \sum_{\sigma \in S_n} sgn(\sigma) \prod_{i=1}^{n} a_{i\sigma(i)}.$$

How do we get this? Well, let's start with the 2×2 case. We can use linearity twice to get:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & 0 \\ c & d \end{vmatrix} + \begin{vmatrix} 0 & b \\ c & d \end{vmatrix}$$
$$= \begin{vmatrix} a & 0 \\ c & 0 \end{vmatrix} + \begin{vmatrix} a & 0 \\ 0 & d \end{vmatrix} + \begin{vmatrix} 0 & b \\ c & 0 \end{vmatrix} + \begin{vmatrix} 0 & b \\ 0 & d \end{vmatrix}$$

In the final sum, the first and last matrices have columns with all 0s, and so those determinants are 0. Therefore, the determinant is

$$\begin{vmatrix} a & 0 \\ 0 & d \end{vmatrix} + \begin{vmatrix} 0 & b \\ c & 0 \end{vmatrix} = ad \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} + bc \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = ad - bc.$$

Now, we know $\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1$ as its formed by switching (transposing) columns 1 and 2 of the identity matrix. So, by properties 1 and 2 of the determinant, its value must be -1.

Now, we can do essentially the same thing for a square matrix of any size. In the $n = 3$ case we split it up into $3^3 = 27$ matrices where for each matrix in the split each row has only one non-zero term. If a column choice is repeated, the the matrix has determinant 0. So, the only ones that matter are when the nonero terms come from different columns. For the 3×3 case there are six such terms:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & & \\ & a_{22} & \\ & & a_{33} \end{vmatrix} + \begin{vmatrix} & a_{12} & \\ & & a_{23} \\ a_{31} & & \end{vmatrix} + \begin{vmatrix} & & a_{13} \\ a_{21} & & \\ & a_{32} & \end{vmatrix}$$

$$+ \begin{vmatrix} a_{11} & & \\ & & a_{23} \\ & a_{32} & \end{vmatrix} + \begin{vmatrix} & a_{12} & \\ a_{21} & & \\ & & a_{33} \end{vmatrix} + \begin{vmatrix} & & a_{13} \\ & a_{22} & \\ a_{31} & & \end{vmatrix}$$

Using linearity we can write this as

$$det(A) = a_{11}a_{22}a_{33} \begin{vmatrix} 1 & & \\ & 1 & \\ & & 1 \end{vmatrix} + a_{12}a_{23}a_{31} \begin{vmatrix} & 1 & \\ & & 1 \\ 1 & & \end{vmatrix} + a_{13}a_{21}a_{32} \begin{vmatrix} & & 1 \\ 1 & & \\ & 1 & \end{vmatrix}$$

$$+ a_{11}a_{23}a_{32} \begin{vmatrix} 1 & & \\ & & 1 \\ & 1 & \end{vmatrix} + a_{12}a_{21}a_{33} \begin{vmatrix} & 1 & \\ 1 & & \\ & & 1 \end{vmatrix} + a_{13}a_{22}a_{31} \begin{vmatrix} & & 1 \\ & 1 & \\ 1 & & \end{vmatrix}$$

Each of the matrices above is a permutation matrix. The first is the identity. The second and third we can get from the identity with two row switches (for the second matrix the row switches are rows 1 and 3, then rows 1 and 2), and the fourth, fifth, and sixth we can get from the identity with one row switch. So, the determinants of the first three are $+1$, and the determinants of the last three are -1. Therefore, the determinant formula for a 3×3 matrix is:

$$a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31}.$$

Now, the sign of a permutation σ is the parity of the number of transpositions required to get it from the identity. It is either even of odd. We say $sgn(\sigma) = \pm 1$ depending on if it requires an even or odd number of tranpositions to get it from the identity. The sign of the first three permutation matrices above are all $+1$, as they require an even number of tranpositions (0, 2, and 2), while the sign of the last three permutation matrices are all -1, as they require an odd number of transpositions (1,1, and 1). If P is a permutation matrix coming from the permutation σ, then $det(P) = sgn(\sigma)$, and from this we get our big formula:

$$det(A) = \sum_{\sigma \in S_n} sgn(\sigma) \prod_{i=1}^{n} a_{i\sigma(i)}.$$

Note that there are $n!$ permutations in S_n, and so as n gets larger this formula gets very big very fast.

Example - Use the big formula to calculate the determinant of the matrix

$$A = \begin{pmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{pmatrix}$$

$det(A) = 126 + 24 + 24$
$- 18 - 112 - 36$
$= 174 - 166$
$= \boxed{8}$

3 Determinants by Cofactors

Finally, we note that if we do some algebra to the formula derived above for the 3×3 determinant we get:

$$det(A) = a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{23}a_{31} - a_{21}a_{33}) + a_{13}(a_{21}a_{32} - a_{22}a_{31}).$$

The terms in parentheses are the determinants

$$\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}, \quad \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}, \quad \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

respectively. The matrices are the matrices we get from A if we eliminate column 1 and row 1, column 2 and row 1, and column 3 and row 1, respectively.

For a matrix A, we denote by M_{ij} the submatrix acquired by eliminating row i and column j. We define the *cofactor* $C_{ij} = (-1)^{i+j} det(M_{ij})$. So, in the 3×3 case above, we have

$$det(A) = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}.$$

And, in fact, this formula holds for any row and for any $n \times n$ matrix A.

$$det(A) = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}.$$

We can also use the formula moving down a column instead of along a row

$$det(A) = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}.$$

Example - For the matrix

$$A = \begin{pmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{pmatrix}$$

calculate the determinant using a cofactor expansion along the first row, and then do it again with a cofactor expansion along the second column.

First row:

$$2(63-9) - 4(28-6) + (-2)(-12-(-18))$$

$$= 108 - 88 - 12 = \boxed{8}$$

Second column:

$$-4(28-6) + 9(14-4) - (-3)(-6-(-8))$$

$$= -88 + 90 + 6 = \boxed{8}$$