Math 2270 - Lecture 26 : The Properties of Determinants

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The lecture covers *section 5.1* from the textbook.

The determinant of a square matrix is a number that tells you quite a bit about the matrix. Formally, it's defined as:

$$det(A) = \sum_{\sigma \in S_n} (-1)^{sgn(\sigma)} \prod_{i=1}^{n} a_{i\sigma(i)}.$$

where $A = (a_{ij})$. Well, that's enlightening, isn't it? That's the last you'll see of that definition today, and instead we'll talk about some properties of the determinant that we'd like it to have. It turns out that with only a few required properties, we can derive a whole bunch.

1 The Three Basic Properties

Recall that the inverse of a 2×2 matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is the matrix

$$A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

This is nice and well defined assuming $ad - bc \neq 0$. If $ad - bc = 0$ then this formula doesn't work, and in fact A is not invertible. For a 2×2 matrix the determinant is the number $ad - bc$. We'll be using the 2×2 example repeatedly as we go over these properties.

The first basic property concerns the determinant of the identity matrix I. In the 2×2 case the identity matrix is

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and the determinant is $1 \times 1 - 0 \times 0 = 1$. In general we'd like for the determinant of the $n \times n$ identity matrix

$$I = \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix}$$

to be 1. In fact, this is the first defining property.

Next, we note that if we switch the rows of a 2×2 matrix:

$$\begin{pmatrix} c & d \\ a & b \end{pmatrix}$$

the determinant is $(bc - ad) = -(ad - bc)$. So, the sign of the determinant is switched. We want this property, that switching the rows changes the sign of the determinant, to be our second defining property.

Finally, we note that the determinant of a 2×2 matrix is a linear function of each row separately. That is to say, if we multiply a row, say the top row, by a number t we get

$$\begin{pmatrix} ta & tb \\ c & d \end{pmatrix}$$

which has determinant $tad - tbc = t(ad - bc)$. So, multiplying a row by t multiplies the determinant by t. As for the other linearity property, when we add two row vectors together we get

$$\begin{pmatrix} a + a' & b + b' \\ c & d \end{pmatrix}$$

the determinant is $ad - bc + a'd - b'c$, which is the sum of the determinants of the matrices

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad \text{and} \qquad \begin{pmatrix} a' & b' \\ c & d \end{pmatrix}.$$

We want this property to hold generally. That is to say, if a row is multiplied by t, the determinant is multiplied by t, and if two rows are added, the determinants are added.

We summarize these three defining properties here.

Property 1 The determinant of the identity matrix, $det(I)$, is 1.

Property 2 Switching two rows changes the sign of the determinant.

Property 3 The determinant is a linear function of each row separately.

It turns out that these three rules completely determine the determinant function! That's kind of amazing, but all the other properties we're going to derive follow from these three rules. In fact, you remember that nasty sum formula at the beginning of the lecutre? It can be proven that it's the only formula that will satisfy the three rules above. Wow!

2 Derived Properties

We'll derive some more properties of the determinant using the three rules above. Some of these properties we'll have as exercises.

Property 4 If two rows of A are equal then $det(A) = 0$.

Exercise - Prove property 4.

Property 5 Subtracting a multiple of one row from another leaves $det(A)$ unchanged.

We can see this is true in the 2×2 case by noting

$$\begin{aligned} det\begin{pmatrix} a & b \\ c - ta & d - tb \end{pmatrix} &= det\begin{pmatrix} a & b \\ c & d \end{pmatrix} - t\left(det\begin{pmatrix} a & b \\ a & b \end{pmatrix}\right) \\ &= det\begin{pmatrix} a & b \\ c & d \end{pmatrix}. \end{aligned}$$

The last equality follows from property 3 (linearity), and the second equality follows from property 4.

Property 6 A matrix with a row of zeros has $det(A) = 0$.

If we multiply the row of zeros by t the matrix is unchanged, and so we must have $det(A) = t \times det(A)$ for all t, which is only true if $det(A) = 0$.

Property 7 If A is triangular then $det(A) = a_{11}a_{22}\cdots a_{nn}$, the product of the diagonal entries.

We can just use elimination to make all the non-diagonal entries 0, and then apply linearity n times.

Property 8 If A is singular then $det(A) = 0$. If A is invertible then $det(A) \neq 0$.

We use elimination to go from A to U. Each step of elimination either leaves the determinant the same, or switches its sign. If A is singular than U has a zero on its diagonal, and so by property 7 the determinant is 0. On the other hand if A is invertible then U has pivots along its diagonal, and the product of non-zero terms is always non-zero.

Property 9 The determinant of AB is $det(A)det(B)$.

The book does this in a very clever way, noting that $det(AB)/det(B)$ satisfies, as a function of A, the three defining properties of the determinant. So, it must be that $det(A) = det(AB)/det(B)$, and therefore $det(AB) = det(A)det(B)$.

In particular, as $AA^{-1} = I$, if A in invertible then $det(A^{-1}) = 1/det(A)$.

Property 10 The transpose A^T has the same determinant as A.

This is actually one property that is more obvious from the formal definition. However, we won't use it. Instead, we can take the LU decomposition of A, writing $PA = LU$. Then using property 9 we get $det(P)det(A) = det(L)det(U)$. As L has 1s all down the diagonal $det(L) = 1$. Now, taking transposes of both sides we get $A^TP^T = U^TL^T$. We must have $det(L^T) = 1$, and as U as U^T are both triangular matrices with the same diagonal we get $det(U) = det(U^T)$ by property 7. So, we get $det(P)det(A) = det(A^T)det(P^T)$. A permutation matrix is formed by permuting the rows of the identity matrix, so its determinant must be either 1 or -1. Also, permutation matrices are orthogonal, so $PP^T = I$, and therefore $det(P)$ and $det(P^T)$ must have the same sign, and therefore be equal. This gives us $det(A) = det(A^T)$, which is what we wanted.

We end with a bit about notation. The determinant of A, $det(A)$, can be written as $|A|$. Also, if you see a matrix with straight lines instead of curved lines enclosing the entries, that means the determinant of the matrix. So, for a 2×2 matrix

$$det(A) = |A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

Example - Calculate

$$\begin{vmatrix} 3 & 4 \\ 2 & 3 \end{vmatrix}.$$