

# Math 2270 - Lecture 24 : Orthogonal Bases

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This lecture and the next will cover *section 4.4* from the textbook.

In section 4.2 we derived the formula for the projection,  $\mathbf{p}$ , of a vector  $\mathbf{b}$  onto the subspace spanned by the columns of a matrix  $A$ . This formula is

$$\mathbf{p} = A(A^T A)^{-1} A^T \mathbf{b}$$

and, in general, to project a vector  $\mathbf{b}$  onto the column space of  $A$  we multiply  $\mathbf{b}$  by the projection matrix  $P$ , where

$$P = A(A^T A)^{-1} A^T.$$

Now,  $P$  can be a real pain to calculate. The part that makes it obnoxious is the  $(A^T A)^{-1}$  term in the middle. Calculating inverses is hard. Wouldn't it be nice if  $A^T A$  were the identity? Well, in the next two lectures we'll talk about when this is the case, and how to, starting with a matrix  $A$ , find a matrix  $Q$  with the same column space, but for which  $Q^T Q = I$ .

The assigned problems for this section are:

*Section 4.4 - 2, 6, 11, 18, 21*

# 1 Orthonormal Vectors

We say a set of vectors  $\mathbf{q}_1, \dots, \mathbf{q}_n$  are *orthogonal* if the dot product of any two distinct vectors from the set is 0. In other words, if  $\mathbf{q}_i^T \mathbf{q}_j = 0$  whenever  $i \neq j$ . If in turn the length of each vector is 1, so  $\mathbf{q}_i^T \mathbf{q}_i = 1$  for all  $i$ , then we say the set is *orthonormal*. Let's write that compactly as a definition.

**Definition** - The vectors  $\mathbf{q}_1, \dots, \mathbf{q}_n$  are *orthonormal* if

$$\mathbf{q}_i^T \mathbf{q}_j = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

*Example* - Prove that a set  $\mathbf{q}_1, \dots, \mathbf{q}_n$  of orthogonal vectors is linearly independent.

We will usually denote a matrix with orthonormal columns as  $Q$ .

What makes a matrix with orthonormal columns easy to work with is  $Q^T Q = I$ . Note that  $Q$  is not required to be square, but of course  $Q^T Q$  will be.

$$Q^T Q = \begin{pmatrix} - & \mathbf{q}_1^T & - \\ - & \mathbf{q}_2^T & - \\ & \vdots & \\ - & \mathbf{q}_n^T & - \end{pmatrix} \begin{pmatrix} | & | & \cdots & | \\ \mathbf{q}_1 & \mathbf{q}_2 & \cdots & \mathbf{q}_n \\ | & | & & | \end{pmatrix} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix} = I.$$

Note that if  $Q$  is square, then  $Q^T Q = I$  means  $Q^T = Q^{-1}$ . So, the transpose is the inverse.

Permutation matrices are matrices that change the order of the components of a vector  $\mathbf{b}$ , but do not change the components themselves. For example, the matrix on the left

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} y \\ z \\ x \end{pmatrix}$$

changes the order  $(x, y, z)$  to  $(y, z, x)$ . This is a permutation matrix, and it's easy to check that it's orthogonal. As square, orthogonal matrices satisfy  $Q^T = Q^{-1}$ , we know the inverse of a permutation matrix is just its transpose.

In general, a permutation matrix is formed by permuting the columns of the identity matrix  $I$ . The columns of the identity matrix are obviously orthogonal, and permuting their positions doesn't change this.

Orthogonal matrices have many nice properties. One is that they leave dot products unchanged. We can see this pretty easily:

$$(Q\mathbf{x}) \cdot (Q\mathbf{y}) = (Q\mathbf{x})^T (Q\mathbf{y}) = \mathbf{x}^T Q^T Q \mathbf{y} = \mathbf{x}^T \mathbf{y} = \mathbf{x} \cdot \mathbf{y}.$$

We determine lengths and angles through dot products, so orthogonal matrices (which produce orthogonal transformations) leave lengths and angles unchanged.

## 2 Projection onto Orthonormal Bases

Suppose we want to project a vector  $\mathbf{b}$  onto the column space of a matrix  $Q$ , where the columns of  $Q$  are orthonormal. Things are so much simpler! In particular, that annoying  $(Q^T Q)^{-1}$  is just the identity. So, we get

$$\begin{aligned}\hat{\mathbf{x}} &= Q^T \mathbf{b}, \\ \mathbf{p} &= Q \hat{\mathbf{x}}, \\ P &= QQ^T.\end{aligned}$$

So much simpler. Let's examine that second equation a little more closely.

$$\mathbf{p} = \left( \begin{array}{c|ccc|c} & & & & \\ \mathbf{q}_1 & \cdots & \mathbf{q}_n & & \\ & & & & \end{array} \right) \begin{pmatrix} \mathbf{q}_1^T \mathbf{b} \\ \vdots \\ \mathbf{q}_n^T \mathbf{b} \end{pmatrix} = \mathbf{q}_1(\mathbf{q}_1^T \mathbf{b}) + \cdots + \mathbf{q}_n(\mathbf{q}_n^T \mathbf{b}).$$

This is a decomposition of  $\mathbf{b}$  into components along the basis vectors  $\mathbf{q}_1, \dots, \mathbf{q}_n$ . Now, in the special case where  $Q$  is square we get  $Q^T Q = I$ , and so  $\mathbf{p} = QQ^T \mathbf{b} = \mathbf{b}$ . So,  $\mathbf{b}$  is already in the column space of  $Q$ , and projection just gives us back  $\mathbf{b}$ . So bloody what?<sup>1</sup>

What's important is the decomposition:

$$\mathbf{b} = \mathbf{q}_1(\mathbf{q}_1^T \mathbf{b}) + \mathbf{q}_2(\mathbf{q}_2^T \mathbf{b}) + \cdots + \mathbf{q}_n(\mathbf{q}_n^T \mathbf{b}).$$

We can decompose  $\mathbf{b}$  into constituent parts, and frequently for the problem we're trying to solve we know how to solve it for these parts. So, we decompose  $\mathbf{b}$  into component parts, solve the problem for the component parts, and then add up all these component solutions to get our solution for  $\mathbf{b}$ . This method is used *all the time* in applied mathematics, most famously for Fourier series, which you'll see next semester if you take differential equations.

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<sup>1</sup>You've got some attitude, mister.

*Example*

Write the vector  $\mathbf{b}$  as a decomposition along the orthonormal basis vectors  $\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3$ .

$$\mathbf{b} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad \mathbf{q}_1 = \frac{1}{3} \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix}, \mathbf{q}_2 = \frac{1}{3} \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}, \mathbf{q}_3 = \frac{1}{3} \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}$$