Math 2270 - Lecture 22 : Projections

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Fall 2012

This lecture covers *section 4.2* from the textbook.

In our last lecture we learned that if A is an $m \times n$ matrix then every vector $\mathbf{x} \in \mathbb{R}^n$ can be written as

$$\mathbf{x} = \mathbf{x}_{row} + \mathbf{x}_{null}$$

where $\mathbf{x}_{row}$ is in the rowspace of A, written $\mathbf{x}_{row} \in \mathbf{C}(A^T)$, and $\mathbf{x}_{null}$ is in the nullspace of A, written $\mathbf{x}_{null} \in \mathbf{N}(A)$. We left relatively unexplored how, given the rowspace or the nullspace, we would actually find these vectors $\mathbf{x}_{row}$ and $\mathbf{x}_{null}$. In order to find these vectors, we need *projections*, and projections are the subject of this lecture. The exercises for section 4.2 will be:

Section 4.2 - 1, 11, 12, 13, 17

1 Projections Onto Lines

When we project a vector $\mathbf{b}$ onto a line[1] we want to find the point on the line closest to the vector $\mathbf{b}$. Call the vector that points from the origin to this closest point $\mathbf{p}$, and note that $\mathbf{p}$ is a vector on this line. The vector that points from $\mathbf{p}$ to $\mathbf{b}$ is called the "error vector", and it will be perpendicular to $\mathbf{p}$, and indeed perpendicular to the line itself.

[1] We'll assume we're always dealing with lines through the origin, as these are the one-dimensional subspaces of $\mathbb{R}^n$.

We want to figure out how we can calculate $\mathbf{p}$ from the vector $\mathbf{b}$ and the line. Well, a line through the origin can be represented as all multiples of a direction vector $\mathbf{a}$, and so we'll describe the line using a direction vector $\mathbf{a}$ parallel to the line. Our vector $\mathbf{p}$ will be some constant multiple of this direction vector $\mathbf{a}$, so $\mathbf{p} = \hat{x}\mathbf{a}$. We want to calculate $\hat{x}$.

Now, the error vector $\mathbf{e} = \mathbf{b} - \mathbf{p}$ will be perpendicular to $\mathbf{a}$, so

$$\mathbf{a} \cdot (\mathbf{b} - \hat{x}\mathbf{a}) = 0.$$

We can rewrite this as

$$\mathbf{a} \cdot \mathbf{b} = \hat{x}\mathbf{a} \cdot \mathbf{a},$$

and solve for $\hat{x}$ to get:

$$\hat{x} = \frac{\mathbf{a} \cdot \mathbf{b}}{\mathbf{a} \cdot \mathbf{a}} = \frac{\mathbf{a}^T\mathbf{b}}{\mathbf{a}^T\mathbf{b}}.$$

I rewrote it using transposes because that's an idea that will generalize beyond just lines. So, to recap, the projection, $\mathbf{p}$, of a vector $\mathbf{b}$ onto the line spanned by the direction vector $\mathbf{a}$ will be:

$$\mathbf{p} = \frac{\mathbf{a}^T\mathbf{b}}{\mathbf{a}^T\mathbf{a}}\mathbf{a}.$$

Example - Calculate the projection of the vector $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ onto the line spanned by the vector $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

2 Projection Onto a Subspace

A line in $\mathbb{R}^m$ is a one-dimensional subspace. Suppose we have a higher-dimensional subspace V, and we want to project a vector **b** onto it. First, we need a description of V, and the best description is a set of basis vectors. So, suppose V is a subspace of $\mathbb{R}^m$ with basis $\mathbf{a}_1, \ldots, \mathbf{a}_n$. The projection of **b** onto V is the vector in V closest to **b**. This projection vector, **p**, will be by definition a linear combination of the basis vectors of V:

$$\mathbf{p} = \hat{x_1}\mathbf{a}_1 + \cdots + \hat{x_n}\mathbf{a}_n.$$

If A is a matrix with column vectors $\mathbf{a}_1, \ldots, \mathbf{a}_n$ and $\hat{\mathbf{x}}$ is a vector with components $\hat{x_1}, \ldots, \hat{x_n}$ then $\mathbf{p} = A\hat{\mathbf{x}}$. The error vector, the vector that points from **p** to **b**, will be $\mathbf{e} = \mathbf{b} - \mathbf{p} = \mathbf{b} - A\hat{\mathbf{x}}$. This error vector will be perpendicular to the subspace V, which is equivalent to being perpendicular to the basis vectors $\mathbf{a}_1, \ldots, \mathbf{a}_n$. Stated mathematically, this is

$$\mathbf{a}_1^T(\mathbf{b} - A\hat{\mathbf{x}}) = 0,$$
$$\vdots$$
$$\mathbf{a}_n^T(\mathbf{b} - A\hat{\mathbf{x}}) = 0.$$

We can write all of the above in matrix form:

$$A^T(\mathbf{b} - A\hat{\mathbf{x}}) = \mathbf{0}.$$

This in turn can be written as

$$A^T A\hat{\mathbf{x}} = A^T\mathbf{b}.$$

If $A^T A$ is invertible we can solve for $\hat{\mathbf{x}}$ to get:

$$\hat{\mathbf{x}} = (A^T A)^{-1} A^T\mathbf{b}.$$

Now, the projection vector $\mathbf{p}$ is the vector $A\hat{\mathbf{x}}$, so

$$\mathbf{p} = A(A^T A)^{-1} A^T\mathbf{b}.$$

In general, to calculate the projection of any vector onto the space W we multiply the vector by the *projection matrix* $P = A(A^T A)^{-1} A^T$.

Wow, that was a lot of work! Let's see how we'd actually use these in an example.

Example - If $A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 6 \\ 0 \\ 0 \end{pmatrix}$ find $\hat{\mathbf{x}}$ and $\mathbf{p}$ and P.

3 So, why is $A^T A$ invertible?

We would be remiss if we ended this section without justifying our assumption that $A^T A$ is invertible. The idea here is that both A and $A^T A$ have the same nullspace. If $\mathbf{x} \in \mathbf{N}(A)$ then $A^T A\mathbf{x} = A^T\mathbf{0} = \mathbf{0}$. So, $\mathbf{x} \in \mathbf{N}(A^T A)$. That's the easy part. Now, suppose $A^T A\mathbf{x} = \mathbf{0}$. This would imply $\mathbf{x}^T A^T A\mathbf{x} = \mathbf{x}^T\mathbf{0} = 0$. Now, $\mathbf{x}^T A^T A\mathbf{x} = (A\mathbf{x})^T(A\mathbf{x}) = ||A\mathbf{x}||^2$. If $||A\mathbf{x}||^2 = 0$ we must have $A\mathbf{x} = \mathbf{0}$, and so $\mathbf{x} \in \mathbf{N}(A)$. Therefore A and $A^T A$ both have the same nullspace.

Now, if the columns of A are linearly independent, then $\mathbf{N}(A) = \mathbf{0}$. Therefore $\mathbf{N}(A^T A) = \mathbf{0}$. As $A^T A$ is square, if it has trivial nullspace it must be invertible. By assumption the column vectors of the matrix A we used to construct our projection matrix are linearly independent, so $A^T A$ has an inverse. Thank goodness!