Math 2270 - Lecture 21 : Orthogonal Subspaces

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This lecture finishes *section 4.1*.

In this lecture we'll delve deeper into the idea of orthogonal complements, and see that the reason they're so important is that if $\mathbf{W}$ is a subspace of a vector space $\mathbf{V}$, then every vector $\mathbf{v} \in \mathbf{V}$ can be written as the sum of a vector from $\mathbf{W}$, and a vector from $\mathbf{W}^{\perp}$. So, $\mathbf{v}$ can be decomposed into a *component* in $\mathbf{W}$, and a component perpendicular to $\mathbf{W}$. The assigned problems for this section are:

Section 4.1 - 6, 7, 9, 21, 24

1 Orthogonal Complements and Decompositions

We recall from the last lecture the definition of the orthogonal complement of a vector subspace.

Definition - If $\mathbf{V}$ is a subspace of a vector space, then the orthogonal complement of $\mathbf{V}$, denoted $\mathbf{V}^{\perp}$, is the set of all vector in the vector space perpendicular to $\mathbf{V}$.

We saw at the end of the last lecture that for an $m \times n$ matrix A the orthogonal complement of the row space $\mathbf{C}(A^T)$ is the nullspace $\mathbf{N}(A)$, and vice-versa.

Now, what's so cool about complements is that we can use them to break down vectors into components. That is to say, for our $m \times n$ matrix A, any vector $\mathbf{x}$ in $\mathbb{R}^n$ can be written as the sum of a component $\mathbf{x}_r$ in the row space, and a component $\mathbf{x}_n$ in the nullspace:

$$\mathbf{x} = \mathbf{x}_r + \mathbf{x}_n.$$

When we multiply A by $\mathbf{x}$, $A\mathbf{x}$, the output will be a vector in the column space of A. In fact, we can view $A\mathbf{x}$ as just a linear combination of the columns of A, where the components of $\mathbf{x}$, $x_1, x_2, \ldots, x_n$ are the coefficients of the linear combination.

What's amazing is that every output vector $\mathbf{b}$ comes from one and only one vector in the row space. The proof is simple. Suppose $A\mathbf{x} = \mathbf{b}$. We write $\mathbf{x} = \mathbf{x}_r + \mathbf{x}_n$, and so

$$\mathbf{b} = A\mathbf{x} = A\mathbf{x}_r + A\mathbf{x}_n = A\mathbf{x}_r + \mathbf{0} = A\mathbf{x}_r.$$

So, we know there is a vector $\mathbf{x}_r$ in the row space such that $A\mathbf{x}_r = \mathbf{b}$. Furthermore, suppose $\mathbf{x}_r'$ is another vector in the row space such that $A\mathbf{x}_r' = \mathbf{b}$. Then we have

$$A(\mathbf{x}_r - \mathbf{x}_r') = A\mathbf{x}_r - A\mathbf{x}_r' = \mathbf{b} - \mathbf{b} = \mathbf{0}.$$

So, $\mathbf{x}_r - \mathbf{x}_r'$ is in the nullspace of A. As its the difference of two vectors in the row space it is also in the row space of A. As the row space and nullspace are orthogonal complements the only vector in both of them is $\mathbf{0}$. So, $\mathbf{x}_r - \mathbf{x}_r' = \mathbf{0}$, and $\mathbf{x}_r = \mathbf{x}_r'$.

What this means is that, as a map from the row space of A to the column space of A, A is invertible. We'll return to this again in about a month.

Example - For $A = \begin{pmatrix} 1 & -1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$ and $\mathbf{x} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$ decompose $\mathbf{x}$ into $\mathbf{x}_r$ and $\mathbf{x}_n$.

$$\vec{N}(A) = \text{span}\left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$$

$$c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

$$c_1 = 1, \quad c_2 = 1$$

So,

$$\begin{pmatrix} 1 \\ -1 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$$ with $\vec{x}_r = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$\vec{x}_n = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

2 Combining Bases from Subspaces

Now, we've stated that if A is an $m \times n$ matrix then any vector in $\mathbb{R}^n$ can be written as the sum of a vector from the row space of A and a vector from the nullspace of A. This is based on the following facts:

1. Any n independent vectors in $\mathbb{R}^n$ must span $\mathbb{R}^n$. So, they are a basis.
2. Any n vectors that span $\mathbb{R}^n$ must be independent. So, they are a basis.

We know that the row space $\mathbf{C}(A^T)$ has dimension r equal to the rank of the matrix A, while the nullspace $\mathbf{N}(A)$ has dimension equal to $n - r$.

If we take a basis for $\mathbf{C}(A^T)$ and a basis for $\mathbf{N}(A)$ then we have n vectors in $\mathbb{R}^n$, and as long as they're linearly independent[1] they span $\mathbb{R}^n$. Suppose $\mathbf{v}_1, \ldots, \mathbf{v}_r$ are a basis for $\mathbf{C}(A^T)$ and $\mathbf{w}_1, \ldots, \mathbf{w}_{n-r}$ are a basis for $\mathbf{N}(A)$. The union of these two sets of vectors is a basis for $\mathbb{R}^n$, and so any vector $\mathbf{x} \in \mathbb{R}^n$ can be written as:

$$\mathbf{x} = c_1\mathbf{v}_1 + \cdots + c_r\mathbf{v}_r + c_{r+1}\mathbf{w}_1 + \cdots + c_n\mathbf{w}_{n-r}.$$

Now, we know

$$c_1\mathbf{v}_1 + \cdots + c_r\mathbf{v}_r \in \mathbf{C}(A^T),$$

and

$$c_{r+1}\mathbf{w}_1 + \cdots + c_{n-r}\mathbf{w}_{n-r} \in \mathbf{N}(A).$$

So, we can write

$$\mathbf{x} = \mathbf{x}_r + \mathbf{x}_n$$

with $\mathbf{x}_r \in \mathbf{C}(A^T)$ and $\mathbf{x}_n \in \mathbf{N}(A)$.

The same idea applies to any vector subspace and its complement.

Example - Why is the union of the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_r$ and $\mathbf{w}_1, \ldots, \mathbf{w}_{n-r}$ a linearly independent set of vectors?

Suppose $\vec{x}_r + \vec{x}_n = \vec{0}$. Then $\vec{x}_r = -\vec{x}_n$. The only vector in both a subspace and its complement is $\vec{0}$, so $\vec{x}_r = \vec{x}_n = \vec{0}$. From here we just use that the $\vec{v}_1, \ldots, \vec{v}_r$ are independent, as are the $\vec{w}_1, \ldots, \vec{w}_{n-r}$.

[1] Proven in the last example.