

Math 2270 - Lecture 20 : Orthogonal Subspaces

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This lecture covers the first part of *section 4.1*.

In this lecture we'll discuss how the idea of orthogonality can be extended from the realm of vectors, where we've already seen it, to the realm of subspaces. The assigned problems for this section are:

Section 4.1 - 6, 7, 9, 21, 24

1 Orthogonal Subspaces

In chapter 1 we said that two vectors $\mathbf{v}$ and $\mathbf{w}$ are *orthogonal*¹ if their dot product, $\mathbf{v} \cdot \mathbf{w}$, is 0. In $\mathbb{R}^2$ or $\mathbb{R}^3$ this matches our geometric understanding of orthogonal, and in higher dimensions the idea still applies, even though we can't visualize it.

¹Or *normal*, or *perpendicular*. These are all synonyms.

Example - Just as a refresher, which of the following vectors are perpendicular?

$$\mathbf{a} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} \quad \mathbf{b} = \begin{pmatrix} 2 \\ 1 \\ 0 \\ -1 \end{pmatrix} \quad \mathbf{c} = \begin{pmatrix} 1 \\ 1 \\ -1 \\ 0 \end{pmatrix}.$$

$$\begin{aligned} \vec{a} \cdot \vec{b} &= 2 + 2 - 4 = 0 \\ \vec{a} \cdot \vec{c} &= 1 + 2 - 3 = 0 \\ \vec{b} \cdot \vec{c} &= 2 + 1 = 3 \end{aligned}$$

So, $\vec{a}$ and $\vec{b}$ are perpendicular, as are $\vec{a}$ and $\vec{c}$, while $\vec{b}$ and $\vec{c}$ are not.

Now, we can extend these definitions to subspaces of a vector space.

Definition - Two subspaces V and W of a vector space are *orthogonal* if every vector $\mathbf{v} \in V$ is perpendicular to every vector $\mathbf{w} \in W$.

As a simple example, in $\mathbb{R}^2$ the span of $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is the set of all vectors of the form $\begin{pmatrix} c \\ 0 \end{pmatrix}$ where c is some real number, while the span of $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is the set of all vectors of the form $\begin{pmatrix} 0 \\ d \end{pmatrix}$ where d is some real number. The dot product $\mathbf{v} \cdot \mathbf{w} = \mathbf{v}^T \mathbf{w}$ of any vector $\mathbf{v}$ in the span of $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ with any vector $\mathbf{w}$ in the span of $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ will be:

$$\begin{pmatrix} c & 0 \end{pmatrix} \begin{pmatrix} 0 \\ d \end{pmatrix} = 0 + 0 = 0.$$

So, the two spans are orthogonal subspaces of $\mathbb{R}^2$. Stated a bit more familiarly, the x -axis and y -axis of the coordinate plane are perpendicular.

For a slightly more complicated example, let's examine an $m \times n$ matrix A . The row space of A is a subspace of $\mathbb{R}^n$, as is the nullspace of A . These two subspaces will, in fact, be orthogonal. This is pretty quick given the definitions of row space, nullspace, and matrix multiplication. Suppose $\mathbf{x}$ is a vector in the nullspace of A . This means $A\mathbf{x} = \mathbf{0}$. From the definition of matrix multiplication we know:

$$A\mathbf{x} = \begin{pmatrix} \text{row1} \\ \text{row2} \\ \vdots \\ \text{rowm} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \mathbf{0}.$$

The dot product of $\mathbf{x}$ with each of the rows must be 0. As the row space is the set of linear combinations of the rows, the dot product of $\mathbf{x}$ with *any* vector in the row space must be 0. So, if $\mathbf{v} \in C(A^T)$ and $\mathbf{w} \in N(A)$ we must have $\mathbf{v}^T \mathbf{w} = 0$. This means the row space and nullspace of A are orthogonal.

Similarly, every vector in the left nullspace of A , $N(A^T)$, is perpendicular to every vector in the column space of A , $C(A)$. So, the column space of A and the left nullspace of A are orthogonal.

Example - Find a vector perpendicular to the row space of the matrix

$$A = \begin{pmatrix} 1 & 3 & 4 \\ 5 & 2 & 7 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 3 & 4 \\ 5 & 2 & 7 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 3 & 4 \\ 0 & -13 & -13 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{aligned} x_3 &= 1 \\ x_1 &= -1 \\ x_2 &= -1 \end{aligned}$$

$$\Rightarrow \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Or any constant multiple of this vector.

2 Orthogonal Complement

If we're given a subspace V of a vector space, and another subspace W orthogonal to it, a natural question to ask is if W is the largest subspace orthogonal to V . Turns out the largest subspace orthogonal to V is unique, and is defined as the *orthogonal complement* of V .

Definition - The *orthogonal complement* of a subspace V contains *every* vector that is perpendicular to V . This orthogonal subspace is denoted by $V^\perp$ (pronounced "V perp").

We saw above that for a matrix A the nullspace $N(A)$ is perpendicular to the row space $C(A^T)$. It turns out the nullspace is in fact the orthogonal complement of the row space. We can see this by noting that if A is an $m \times n$ matrix both the row space and the nullspace are subspaces of $\mathbb{R}^n$. The dimension of the nullspace is $n - r$, where r is the rank of A , which is also the dimension of the row space. If x were a vector orthogonal to the row space, but not in the nullspace, then the dimension of $C(A^T)^\perp$ would be at least $n - r + 1$. But this would be too large for both $C(A^T)$ and $C(A^T)^\perp$ to fit in $\mathbb{R}^n$! So, the nullspace is the largest subspace perpendicular to the row space, and we have $C(A^T)^\perp = N(A)$. Similarly $N(A)^\perp = C(A^T)$.

Example - Calculate $V^\perp$ if V is the vector space given by:

$$V = \text{span} \left\{ \begin{pmatrix} 1 \\ 2 \\ 5 \\ 4 \end{pmatrix}, \begin{pmatrix} 3 \\ 7 \\ 3 \\ 12 \end{pmatrix} \right\}.$$

$$\begin{pmatrix} 12 & 5 & 4 \\ 3 & 7 & 12 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 2 & 5 & 4 \\ 0 & 1 & -12 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 29 & 4 \\ 0 & 1 & -12 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 29 & 4 \\ 0 & 1 & -12 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \quad \begin{matrix} x_3 = 1 \\ x_4 = 0 \\ x_2 = 12 \\ x_1 = -29 \end{matrix} \quad \begin{pmatrix} -29 \\ 12 \\ 0 \\ 1 \end{pmatrix}$$

$$\begin{matrix} x_3 = 0 \\ x_4 = 1 \\ x_2 = 0 \\ x_1 = -4 \end{matrix} \quad \begin{pmatrix} -4 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\vec{V}^\perp = \text{span} \left\{ \begin{pmatrix} -29 \\ 12 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -4 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

