Math 2270 - Lecture 19: The Four Subspaces

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This lecture covers section 3.6 of the textbook.

In this lecture we introduce a new fundamental subspace of a matrix, the "left nullspace", and discuss the various dimensions of the four fundamental subspaces, and how they fit together. The problems for this section are:

Section 3.6 - 1, 3, 5, 11, 24

1 A New Subspace

So far we've seen and discussed three subspaces of an $m \times n$ matrix:

1. The *row space*, $\mathbf{C}(A^T)$, which is in $\mathbb{R}^n$.

2. The *column space*, $\mathbf{C}(A)$, which is in $\mathbb{R}^m$.

3. The *nullspace*, $\mathbf{N}(A)$, which is in $\mathbb{R}^n$.

Today we add to these a fourth subspace, which is the similar to the nullspace. We call this the "left nullspace", and it's the nullspace of A^T:

4. The *left nullspace*, $\mathbf{N}(A^T)$, which is in $\mathbb{R}^m$.

1

The left nullspace is the space of all vectors $\mathbf{y}$ such that $A^T\mathbf{y} = \mathbf{0}$. It can equivalently be viewed as the space of all vectors $\mathbf{y}$ such that $\mathbf{y}^TA = \mathbf{0}$. Thus the term "left" nullspace.

Now, the rank of a matrix is defined as being equal to the number of pivots. The rank of a matrix is also equal to the dimension of *both* the column space and the row space.

We calculate the number of pivots by reducing the matrix to reduced row echelon form (or just echelon form, either works). When we do this some of the rows are non-zero, and some of the rows are zero. The non-zero rows, by construction, are all linearly independent, and in fact form a basis for the row space. Each row, again by construction, has one and only one pivot. Therefore, the rank of a matrix is equal to the dimension of its row space.

Now, the column space is a little more complicated, but only just a little. If we get our matrix into reduced row echelon form, the pivot columns *do not*, necessarily, form a basis for our column space. However, the corresponding columns in the original matrix *do*. Each pivot column has one and only one pivot, and therefore the rank of a matrix is equal to the dimension of its column space.

Finally, a basis for the nullspace of a matrix A is given by the set of "special solutions". There is a special solution for each free variable, and each free variable corresponds with a free column. If A is an $m \times n$ matrix it has n columns, and if it has rank r then r of these columns are pivot columns. If a column is not a pivot column it's a free column, and so the number of free columns is $n - r$. Each free column corresponds with a special solution, and the special solutions are a basis for the nullspace. So, the dimension of the nullspace will be $n - r$.

Now, the rank of A^T is the same as the rank of A, and the left nullspace is the nullspace of the $n \times m$ matrix A^T. So, by the same reasoning we used for the nullspace, the dimension of the left nullspace will $m - r$.

We can summarize these results. For an $m \times n$ matrix A of rank r:

1. The dimension of the column space $\mathbf{C}(A)$ is r.

2. The dimension of the row space $\mathbf{C}(A^T)$ is r.

3. The dimension of the nullspace $\mathbf{N}(A)$ is $n - r$.

4. The dimension of the left nullspace $\mathbf{N}(A^T)$ is $m - r$.

Example - Find a basis for each of the four subspaces associated with A:

$$A = \begin{pmatrix} 1 & 2 & 4 \\ 2 & 4 & 8 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 2 & 4 \\ 2 & 5 & 8 \end{pmatrix}.$$

$A = \begin{pmatrix} 1 & 2 & 4 \\ 2 & 4 & 8 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 2 & 4 \\ 0 & 0 & 0 \end{pmatrix}$ Basis for row space $\begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}$

Basis for column space $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$\begin{pmatrix} 1 & 2 & 4 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$\begin{array}{ll} x_2 = 1 & x_2 = 0 \\ x_3 = 0 & x_3 = 1 \\ x_1 = -2 & x_1 = -4 \end{array}$ Basis for nullspace $\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -4 \\ 0 \\ 1 \end{pmatrix}$

$\begin{pmatrix} 1 & 2 \\ 2 & 4 \\ 4 & 8 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 2 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$ Basis for left nullspace $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$

$B = \begin{pmatrix} 1 & 2 & 4 \\ 2 & 5 & 8 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 2 & 4 \\ 0 & 1 & 0 \end{pmatrix}$ Basis for rowspace $\begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

Basis for column space $\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 5 \end{pmatrix}$

$\begin{pmatrix} 1 & 2 & 4 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$\begin{array}{l} x_3 = 1 \\ x_2 = 0 \\ x_1 = -4 \end{array}$ Basis for nullspace $\begin{pmatrix} -4 \\ 0 \\ 1 \end{pmatrix}$

$\begin{pmatrix} 1 & 2 \\ 2 & 5 \\ 4 & 8 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 2 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$ left nullspace is $\vec{0}$, basis is empty set $\emptyset$.

3