Math 2270 - Lecture 18: Basis and Dimension

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This lecture finishes up section 3.5 from the textbook.

In this lecture we finish up our discission of basis vectors and formally introduce the concept of dimension. We'll actually give an honest proof of something here, which is something that has been lacking in the class so far. Expect to see *many* more proofs in future math classes. Also, we'll end with a bit of a digression that's not in the textbook, but I think is important that you see. Just as a reminder, the assigned problems for section 3.5 are:

Section 3.5 - 1, 2, 3, 20, 28

1 Basis and Dimension

We recall from the end of our last lecture the definition of a basis:

Definition - A *basis* for a vector space is a set of linearly independent vectors that span the vector space.

We saw that for our example matrix

$$A = \begin{pmatrix} 1 & 0 & 3 \\ 2 & 1 & 5 \\ 1 & 0 & 3 \end{pmatrix}$$

any choice of two columns is a basis for the column space of A. This example underscores the fact that a basis for a vector space is not, in general, unique. However, it also hints at something deeper. Each of these bases consists of two vectors. We might ask if all bases for $\mathbf{C}(A)$ consist of two vectors. The answer is yes. In fact, if a vector space has a finite basis, all bases for that vector space have the same number of vectors.

This isn't obvious, so let's prove it.

Theorem - If $\mathbf{v}_1, \ldots, \mathbf{v}_m$ and $\mathbf{w}_1, \ldots, \mathbf{w}_n$ are both bases for the same vector space, then $m = n$.

Proof - Suppose that $n > m$. We know the **v**s are a basis, and so we can write each $\mathbf{w}_i$ as:

$$\mathbf{w}_i = a_{1i}\mathbf{v}_1 + \cdots + a_{mi}\mathbf{v}_m$$

We note that if we define a matrix W as the matrix whose columns are the vectors $\mathbf{w}_i$ then we have

$$W = \begin{pmatrix} \mathbf{w}_1 & \cdots & \mathbf{w}_n \end{pmatrix} = \begin{pmatrix} \mathbf{v}_1 & \cdots & \mathbf{v}_m \end{pmatrix} \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} = VA$$

We know $n > m$, so A is a "short and wide" matrix. We therefore know $A\mathbf{x} = \mathbf{0}$ must have a nontrivial solution! This means the vectors $\mathbf{w}_1, \ldots, \mathbf{w}_n$ must be linearly dependent. This is a contradiction. Identical reasoning proves we cannot have $m < n$, so the only possibility is $m = n$.[1]

So, the number of vectors in a basis of a vector space will always be the same. We say that this number if the *dimension* of the vector space, and we just proved this dimension is well-defined.

Definition - The dimension of a space is the number of vectors in every basis.

[1]Wasn't that cool?!

Example - Find a basis for the plane $x - 2y + 3z = 0$ in $\mathbb{R}^3$. Then find a basis for the intersection of that plane with the xy plane. Then find a basis for all vectors perpendicular to the plane.

$$\begin{pmatrix} 1 & -2 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \vec{0}$$

y, z are free variables.

$y = 1, z = 0, x = 2$

$$\vec{s}_1 = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$$

$y = 0, z = 1, x = -3$

$$\vec{s}_2 = \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix}$$

Basis: $\left\{ \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix} \right\}$

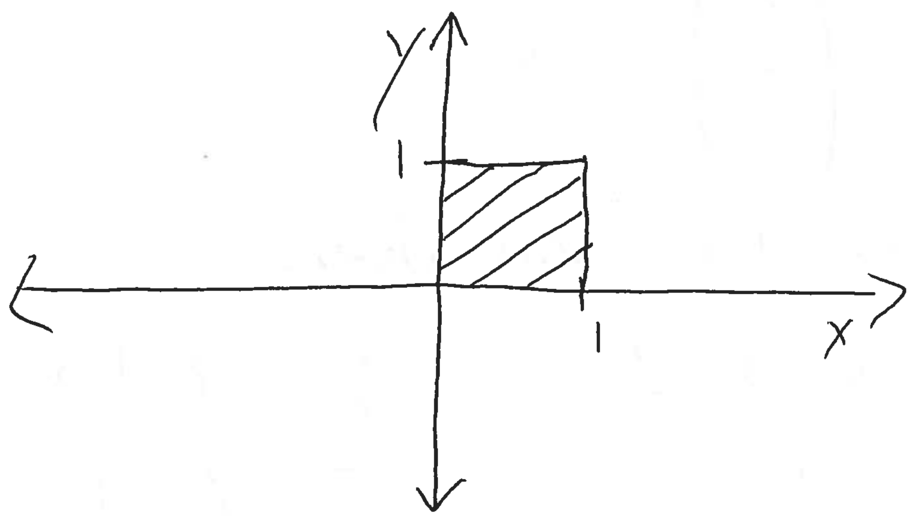
Intersection with xy-plane is $z = 0$.

So, basis: $\left\{ \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \right\}$

The normal vector $\begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$ is a basis for all vectors perpendicular to the plane.

2 A Digression on Dimension

Intuitively, we probably think of dimension as the number of numbers you need in order to specify a particular point in a set. For example, take the unit square in $\mathbb{R}^2$:



This is obviously two dimensional, and we can specify any point with its coordinates (x, y). Two numbers, two dimensional, makes sense.

Our intuition is wrong. We can actually specify any point in this set uniquely with only *one* number! Suppose we write x and y as their decimal expansion:

$$x = .x_1x_2x_3\cdots$$
$$y = .y_1y_2y_3\cdots$$

where each x_i, y_i is a number between 0 and 9. Well, we can map these two numbers to the single number

$$r = .x_1y_1x_2y_2\cdots$$

Similarly, if we're given a number r between 0 and 1 we can write it out in its decimal expansion

$$r = .r_1 r_2 r_3 \cdots$$

and map this to the two numbers

$$x = .r_1 r_3 \cdots$$
$$y = .r_2 r_4 \cdots$$

This gives us a bijection between the points in the unit square, and the points on the line segment between 0 and 1. But, the line segment between 0 and 1 is obviously one dimensional! So, you in fact *don't* need two distinct numbers to specify a point in the unit square. You just need one. It turns out that the notion of dimension in mathematics is much more subtle and interesting that it may at first appear.[2]

[2]The above proof, and the ideas behind the proof, were part of Georg Cantor's groundbreaking work in set theory at the end of the 19th century.