Math 2270 - Lecture 17: Linear Independence

Dylan Zwick

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This lecture covers the first part of section 3.5 from the textbook.

In this lecture we will, finally, fully introduce the idea of linearly independent vectors, along with the ideas of spanning and basis. The assigned problems for this section are:

Section 3.5 - 1, 2, 3, 20, 28

1 Linear Independence

Let's begin, as we so often do, with an example. Take the matrix

$$A = \begin{pmatrix} 1 & 0 & 3 \\ 2 & 1 & 5 \\ 1 & 0 & 3 \end{pmatrix}$$

Now, the column space of A is, by definition, all vectors that can be written as a linear combination of the column vectors of A. But, are all of the columns necessary? That is to say, could we use fewer columns, and still get the same column space? To answer this question, we need to figure out the rank of the matrix A. Getting A into reduced row echelon form we have

$$R = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

There are only two pivots, and only two pivot columns, namely columns 1 and 2. Column 3 is a free column, and is in fact equal to 3 times column 1 minus column 2. What this means is that any vector that can be written as a linear combination of the three columns:

$$\mathbf{b} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$$

can in fact be written as a linear combination of just the first two columns

$$\mathbf{b} = (c_1 + 3c_3)\mathbf{v}_1 + (c_2 - c_3)\mathbf{v}_2.$$

So, we only need the first two vectors to span the entire column space. The third vector is unnecessary.

The column vector $\mathbf{v}_3$ can be written as a linear combination of the other two vectors

$$\mathbf{v}_3 = 3\mathbf{v}_1 - \mathbf{v}_2$$

but there's nothing special about $\mathbf{v}_3$ in this case. We can also write $\mathbf{v}_2$ as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_3$

$$\mathbf{v}_2 = 3\mathbf{v}_1 - \mathbf{v}_3.$$

The important idea is not that one vector can be written as a linear combination of the other two, but that we have the equation

$$3\mathbf{v}_1 - \mathbf{v}_2 - \mathbf{v}_3 = \mathbf{0}.$$

Equations of this form are at the heart of the idea of linear independence. Note that for any set of vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ in a vector space there will be a set of coefficients $c_1, \ldots, c_n$ such that

$$c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 + \cdots + c_n \mathbf{v}_n = \mathbf{0}.$$

These coefficients will be $c_1 = c_2 = \cdots = c_n = 0$. This is called the *trivial* solution. If there is a choice of coefficients where not all the coefficients are 0, then we have a horse of a different color.

Definition - The set of vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ is *linearly independent* if and only if the only coefficients that satisfy the relation

$$c_1 \mathbf{v}_1 + \cdots + c_n \mathbf{v}_n = \mathbf{0}$$

are $c_1 = c_2 = \cdots = c_n = 0$. If a sequence of vectors is not linearly independent, we say they are *linearly dependent*.[1]

So, for example, the columns of our matrix A are linearly dependent. The columns of the identity matrix I, in any number of dimensions, are linearly independent.

Example - If any vector $\mathbf{v}_i$ in a sequence of vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ is the zero vector $\mathbf{v}_i = \mathbf{0}$ then why is the sequence of vectors linearly dependent.

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[1]No surprises there.

2 Spanning and Basis

For our example matrix A we found that the entire column space could be written as a linear combinations of just the first two vectors. We can say this more mathematically by saying that the first two vectors *span* the column space. However, it's also, by definition, the case that the linear combinations of the column vectors of A fill the column space. So, the three column vectors also *span* the column space.

Definition - A set of vectors **spans** a space if their linear combinations fill the space.

Now, we just mentioned that both the first two columns of A and all three columns of A span the column space of A. The first two columns are linearly independent, while the three columns are linearly dependent. If a set of vectors spans a space *and* are linearly independent then they're a special set we call a *basis*.

Definition - A *basis* for a vector space is a sequence of vectors that are linearly independent and that span the entire vector space.

We've seen that for any matrix A we can write all the free columns as linear combinations of the pivot columns. It will also be the case that the pivot columns are linearly independent. So, the pivot columns are a basis for the column space of a matrix. Note that this basis will, in general, not be unique.

Example - Find a basis for the column space of the matrix

$$\begin{pmatrix} 1 & 3 & 2 \\ 0 & 1 & 1 \\ 1 & 3 & 2 \end{pmatrix}$$