

Math 2270 - Lecture 16: The Complete Solution to $A\mathbf{x} = \mathbf{b}$

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This lecture covers *section 3.4* of the textbook.

In this lecture we extend our previous lectures about the nullspace of solutions to $A\mathbf{x} = \mathbf{0}$ to a discussion of the complete set of solutions to the equation $A\mathbf{x} = \mathbf{b}$. The assigned problems for this section are:

Section 3.4 - 1, 4, 5, 6, 18

Up to this point in our class we've learned about the following situations:

1. If A is a square matrix, then if A is invertible every equation $A\mathbf{x} = \mathbf{b}$ has one and only one solution. Namely, $\mathbf{x} = A^{-1}\mathbf{b}$.
2. If A is not invertible, then $A\mathbf{x} = \mathbf{b}$ will have either no solution, or an infinite number of solutions.
3. If $\mathbf{b} = \mathbf{0}$ then the set of all solution to $A\mathbf{x} = \mathbf{0}$ is called the *nullspace* of A , and we've learned how to find all vectors in this nullspace as linear combinations of "special solutions".

Today we're going to extend these ideas to solving the general problem $A\mathbf{x} = \mathbf{b}$.

1 Finding a Particular Solution

Let's begin with an example. Suppose we're given a system of equations in matrix form:

$$\begin{pmatrix} 1 & 3 & 0 & 2 \\ 0 & 0 & 1 & 4 \\ 1 & 3 & 1 & 6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 6 \\ 7 \end{pmatrix}$$

We want to find all solution vectors $\mathbf{x}$ that satisfy the above equation. We can find one solution vector by creating an augmented matrix $(A \ \mathbf{b})$ where we attach the vector $\mathbf{b}$ to the matrix A as a final column on the right. In our example this would be:

$$(A \ \mathbf{b}) = \begin{pmatrix} 1 & 3 & 0 & 2 & 1 \\ 0 & 0 & 1 & 4 & 6 \\ 1 & 3 & 1 & 6 & 7 \end{pmatrix}$$

We then reduce this augmented matrix to reduced row echelon form

$$\begin{pmatrix} 1 & 3 & 0 & 2 & 1 \\ 0 & 0 & 1 & 4 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The final row is all 0s. For the matrix A we have that row 3 is equal to row 1 plus row 2. If this is not also the case for the vector $\mathbf{b}$, then there would be no solution to our system. As things are, we can find a solution as just a combination of our pivot columns. In this case that solution will be $x_1 = 1, x_3 = 6$, with the free variables all set to 0, so $x_2 = x_4 = 0$. If there is a solution, you will always be able to find it with the free variables set to 0.

So, one solution is

$$\mathbf{x}_p = \begin{pmatrix} 1 \\ 0 \\ 6 \\ 0 \end{pmatrix}$$

But, is this the only solution? No. In general, suppose you have two solutions $\mathbf{x}_1$ and $\mathbf{x}_2$ to $A\mathbf{x} = \mathbf{0}$. We can write $\mathbf{x}_2 = \mathbf{x}_1 + (\mathbf{x}_2 - \mathbf{x}_1)$. The vector $\mathbf{x}_2 - \mathbf{x}_1$ will be in the nullspace of A , as

$$A(\mathbf{x}_2 - \mathbf{x}_1) = A\mathbf{x}_2 - A\mathbf{x}_1 = \mathbf{b} - \mathbf{b} = \mathbf{0}.$$

So, if $\mathbf{x}_p$ is a solution to $A\mathbf{x} = \mathbf{0}$, any other solution can be written as the sum of $\mathbf{x}_p$ and a vector in the nullspace. On the other hand, if $\mathbf{x}_n$ is in the nullspace of A , then

$$A(\mathbf{x}_p + \mathbf{x}_n) = A\mathbf{x}_p + A\mathbf{x}_n = \mathbf{b} + \mathbf{0} = \mathbf{b}$$

So, the set of all solutions to $A\mathbf{x} = \mathbf{b}$ is the set of all vectors $\mathbf{x}_p + \mathbf{x}_n$, where $\mathbf{x}_p$ is *any* particular solution¹, and $\mathbf{x}_n$ is a vector in $N(A)$.

Returning to our example, the reduced row echelon form of A is

$$R = \begin{pmatrix} 1 & 3 & 0 & 2 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

From this we can see that the two “special solutions” to $A\mathbf{x} = \mathbf{0}$ will be the vectors

$$\mathbf{s}_1 = \begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad \mathbf{s}_2 = \begin{pmatrix} -2 \\ 0 \\ -4 \\ 1 \end{pmatrix}.$$

¹You just pick one.

Now, the set of all linear combinations of the special solutions (the *span* of the special solutions) is the entire nullspace. So, the complete solution to $Ax = b$ for our example will be all vectors of the form

$$\begin{pmatrix} 1 \\ 0 \\ 6 \\ 0 \end{pmatrix} + c_1 \begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} -2 \\ 0 \\ -4 \\ 1 \end{pmatrix}$$

The general idea of this lecture is that to find the total solution (the set of all solutions) to the equation $Ax = b$ we first find a particular solution where all the free variables are 0, and then determine the nullspace of A by finding all special solutions. The complete solution will be all vectors that can be written as $x_p + x_n$, where x_p is our particular solution, and x_n is a vector in the nullspace.

Example - Find the complete solution to the sequence of equations

$$\begin{aligned} x + 3y + 3z &= 1 \\ 2x + 6y + 9z &= 5 \\ -x - 3y + 3z &= 5 \end{aligned}$$

$$\begin{pmatrix} 1 & 3 & 3 & 1 \\ 2 & 6 & 9 & 5 \\ -1 & -3 & 3 & 5 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 3 & 3 & 1 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 6 & 6 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 3 & 3 & 1 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} 1 & 3 & 3 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 3 & 0 & -2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \vec{x}_p = \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$$

$$\begin{aligned} x_2 &= 1 \\ x_3 &= 0 \\ x_1 &= -3 \end{aligned} \quad \vec{x}_{s_1} = \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix}$$

Free

$$\vec{x} = \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} + c_1 \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix}$$

2 Rectangular Matrices

We've already studied square matrices in depth. Now, let's look a little deeper into rectangular matrices. Suppose we have a matrix with at least as many rows as columns. So, if A is an $m \times n$ matrix, we assume $m \geq n$. The book calls such a matrix "tall and thin". Suppose we're given an equation of the form $A\mathbf{x} = \mathbf{b}$, where A is tall and thin. For example

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ -2 & -3 \end{pmatrix}$$

For matrices of this type it will not be the case, in general, that $A\mathbf{x} = \mathbf{b}$ has a solution. For the matrix A above if we take our augmented matrix and get it into reduced row echelon form

$$\begin{pmatrix} 1 & 1 & b_1 \\ 1 & 2 & b_2 \\ -2 & -3 & b_3 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 2b_1 - b_2 \\ 0 & 1 & b_2 - b_1 \\ 0 & 0 & b_1 + b_2 + b_3 \end{pmatrix}$$

We see that for there to be any solution we must have $b_1 + b_2 + b_3 = 0$. If this is the case, then there is only one solution, namely $x_1 = 2b_1 - b_2$ and $x_2 = b_2 - b_1$. It's important to note here that every column of A is a pivot column. In general, we say that if a matrix has full column rank, then the rank of the matrix, r , is equal to the number of columns in the matrix, n .

All of the following are equivalent criteria for a matrix A to have full column rank:

1. All columns of A are pivot columns.
2. There are no free variables or special solutions.
3. The nullspace $\mathbf{N}(A)$ contains only the zero vector $\mathbf{x} = \mathbf{0}$.
4. If $A\mathbf{x} = \mathbf{b}$ has a solution (it might not) then it has only *one solution*.

Note that all matrices with full column rank are “tall and thin”.²

Now let’s take a look at the other type of rectangular matrix. Namely, one with at least as many columns as rows. Such a matrix is referred to as “short and wide” in the textbook. Suppose further than the rank of the matrix is the same as the number of rows, so the matrix has “full row rank”. Said more mathematically, if the matrix is an $m \times n$ matrix with rank r we assume $r = m$.

For example, the matrix

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & -1 \end{pmatrix}$$

has reduced row echelon form

$$\begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & -2 \end{pmatrix}$$

So, the rank of A is 2, and in reduced row echelon form, every row has a pivot. Now, *any* equation $A\mathbf{x} = \mathbf{b}$ for a matrix with full row rank will have a solution, and possibly an infinite number of solutions. In fact, all of the following properties for an $m \times n$ matrix mean the matrix has full row rank ($r = m$):

1. All rows have pivots, and R has no zero rows.
2. $A\mathbf{x} = \mathbf{b}$ has a solution for every right side $\mathbf{b}$.
3. The column space is the whole space $\mathbb{R}^m$
4. There are $n - r = n - m$ special solution in the nullspace of A .

Note that for a matrix to have full row rank, it must be short and wide.

Finally, we note that if a square matrix A is invertible, it has both full column rank and full row rank. This means, among other things, that there

²Also note that, according to our definition, a square matrix is tall and thin.

is one and only one solution to $Ax = b$, which confirms what we already knew.³

Example - Find the complete solution $Ax = b$ for

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \end{pmatrix} \quad b = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 1 & b_1 \\ 1 & -1 & 1 & b_2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 1 & b_1 \\ 0 & -2 & 0 & b_2 - b_1 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} 1 & 1 & 1 & b_1 \\ 0 & 1 & 0 & \frac{b_1}{2} - \frac{b_2}{2} \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 1 & \frac{b_1}{2} + \frac{b_2}{2} \\ 0 & 1 & 0 & \frac{b_1}{2} - \frac{b_2}{2} \end{pmatrix}$$

$$\vec{x}_p = \begin{pmatrix} \frac{b_1}{2} + \frac{b_2}{2} \\ \frac{b_1}{2} - \frac{b_2}{2} \\ 0 \end{pmatrix}$$

~~Multispace~~
 ~~$x_2 = 0$~~

$$\vec{x}_{s_1} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

$$x_3 = 1$$

$$x_2 = 0$$

$$x_1 = -1$$

$$\vec{x} = \begin{pmatrix} \frac{b_1 + b_2}{2} \\ \frac{b_1 - b_2}{2} \\ 0 \end{pmatrix} + c_1 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

³That one and only one solution is given by the inverse of A .

