

Math 2270 - Lecture 15: The Rank

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This lecture covers the *rank* of a matrix. It corresponds with section 3.3 of the textbook. The assigned problems for this section are:

Section 3.3 - 1, 3, 17, 19, 22

I mentioned the word “rank” at one point in the last lecture, when referring to the rank-nullity theorem. In this lecture we’ll discuss what that means, and how, for a given matrix, it can be determined.

1 The Definition of Rank

Let’s cut the small talk and just get straight to the definition:

Definition - The rank of a matrix A is the number of pivots.

We’ve learned a process, an algorithm, called *elimination* by which we reduce a matrix A into successively simpler forms, and then use these forms to answer questions about the action of A on a vector \mathbf{x} . We’ve seen how to use elimination to solve an equation of the form $A\mathbf{x} = \mathbf{b}$, when there is a unique solution, how to use elimination to calculate the inverse, A^{-1} , of a matrix A , when there is an inverse, and how to use elimination to calculate the *nullspace* of a matrix A , which is the set of all solutions \mathbf{x} to $A\mathbf{x} = \mathbf{0}$.

This process of elimination involves determining pivots and pivot columns. For a given matrix A , square or not, some columns will be pivots and others will be free. If there are free columns, we’ve learned how to find “special solutions” that satisfy the equation $A\mathbf{x} = \mathbf{0}$.

So far we haven't said anything new. So much for not beating around the bush. What we haven't explained yet is that every free column in a matrix is, in fact, a combination of earlier pivot columns, and the special solutions tell you what this combination is!

Let's look at an example. Consider the matrix

$$A = \begin{pmatrix} 1 & 1 & 2 & 4 \\ 1 & 2 & 2 & 5 \\ 1 & 3 & 2 & 6 \end{pmatrix}$$

Elimination tells us to subtract row 1 from row 2, then row 1 from row 3, and finally 2 times the new row 2 from row 3. At the end of this we'll get a matrix in upper-triangular form:

$$\begin{pmatrix} 1 & 1 & 2 & 4 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

There are two pivot columns, columns 1 and 2, and two free columns, columns 3 and 4. If we set the corresponding free variables to be $x_3 = 1$ and $x_4 = 0$ we get $x_2 = 0$ and $x_1 = -2$, giving us the first special solution

$$\mathbf{s}_1 = \begin{pmatrix} -2 \\ 0 \\ 1 \\ 0 \end{pmatrix}.$$

If we set $x_3 = 0$ and $x_4 = 1$ we get $x_2 = -1$ and $x_1 = -3$, giving us the special solution

$$\mathbf{s}_2 = \begin{pmatrix} -3 \\ -1 \\ 0 \\ 1 \end{pmatrix}$$

The first special solution, if read correctly, tells us that column 3 is in fact 2 times column 1. The second special solution tells us that column 4 is 3 times column 1 plus column 2. So, both free columns are, in fact, combinations of the pivot columns.

Now, why is this? Well, I think it becomes much more clear if we step through what the special solution means. Let's take a look at \mathbf{s}_2 . We've calculated \mathbf{s}_2 so that $U\mathbf{s}_2 = \mathbf{0}$. Writing this out long form we get:

$$\begin{pmatrix} 1 & 1 & 2 & 4 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} -3 \\ -1 \\ 0 \\ 1 \end{pmatrix} = -3 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 4 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

We can do some algebra to the last equality to get

$$\begin{pmatrix} 4 \\ 1 \\ 0 \end{pmatrix} = 3 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix},$$

which is how we write the fourth column of U as a linear combination of the first two (pivot) columns. Well, OK, that's kind of obvious for U , but what about for our original matrix A ? Well, remember, the nullspace of U is *the same* as the nullspace of A . So, this means that if \mathbf{x} satisfies $U\mathbf{x} = \mathbf{0}$, then that same \mathbf{x} must satisfy $A\mathbf{x} = \mathbf{0}$. What this means is that any relation among the columns of U , must also hold for the columns of A . We can demonstrate this by noting

$$\begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} = 3 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}.$$

So, the pivot columns of A are exactly the same as the pivot columns of U , and they'd be exactly the same as the pivot columns of R were we to keep reducing. The free columns can be written as linear combinations of the pivot columns, and the special solutions tells us what these linear combinations are. This paragraph is, really, the crux of this lecture.

Example - Figure out the pivot columns and free columns of the matrix

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 3 & 6 & 3 \\ 4 & 8 & 5 \end{pmatrix}$$

and write each free column as a linear combination of the pivot columns.

2 The Rank and the Column Space

All the free columns of A can be written as linear combinations of the pivot columns. Importantly, *none* of the pivot columns can be written as linear combinations of the others. All the pivot columns are what is called *linearly independent*. In fact, the pivot columns of a matrix A are a *basis* for the column space of A , and they will *span* this column space. The rank of the matrix A will be the *dimension* of the column space.

If you tell a mathematician that the rank of a matrix is the number of pivots, he'll look at you like you're an ignorant peasant.¹ The mathematician's definition of rank is that it's the dimension of the column space. We'll be delving into what that means, and what all the emphasized terms in the last paragraph mean, later on, but I wanted you to see these terms now just in case you become too enamored with all this talk of "pivots" and develop an unfortunate accent that might hinder you later in your mathematical life.

¹Assuming he's one of those social mathematicians who looks at you.