Math 2270 - Lecture 14: Hunting the Nullspace

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This is the second of two lectures covering the nullspace of a matrix A.

In this lecture we move on from the "what" of a nullspace, and onto the "how". In particular, we'll talk about how we find the nullspace of a matrix A by finding a set of special vectors, a "basis", whose span is the entire nullspace of A.

1 Pivots and Frees

Let's take a look at the matrix

$$A = \begin{pmatrix} 1 & 1 & 2 & 3 \\ 2 & 2 & 8 & 10 \\ 3 & 3 & 10 & 13 \end{pmatrix}$$

We want to find the vectors $\mathbf{x} \in \mathbb{R}^4$ such that $A\mathbf{x} = \mathbf{0}$. To do this, we perform forward elimination to transform A into what is not generally but should be called "staircase form". The first step in doing this is that we take our first row and first column, and note that the entry $a_{11} \neq 0$, and therefore is a pivot. We use this pivot to eliminate all the entries below it:

$$A \rightarrow \begin{pmatrix} 1 & 1 & 2 & 3 \\ 0 & 0 & 4 & 4 \\ 0 & 0 & 4 & 4 \end{pmatrix}$$

We now move onto row 2 and column 2. The diagonal entry a_{22} is, in fact, 0. No, problem, we just exchange row 2 with a row beneath it that does not have a 0 in column 2. Except... each row (which in this case is just row 3) does. Ruh roh! Well, if A were square this would mean it's not invertible. But, here A isn't square, so of course it's not invertible. The trick here is to keep calm and carry on. We stay on row 2, but move onto column 3. Column 3 has a non-zero entry in row 2, $a_{23} = 4$, so *that* will be our pivot for row 2. We use it to eliminate all the entries below it[1]

$$A \rightarrow \begin{pmatrix} 1 & 1 & 2 & 3 \\ 0 & 0 & 4 & 4 \\ 0 & 0 & 4 & 4 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 2 & 3 \\ 0 & 0 & 4 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Our third row is the last row, and it's a whole lotta nothing, so there are no pivots there. So, what we have here is there are two pivot columns, columns 1 and 3, and two free columns, columns 2 and 4. The number of free columns is the dimension of the null space! And we use these free columns to generate our "special" solutions. These special solutions will be vectors that span the nullspace.

The free columns are 2 and 4, so the free variables are x_2 and x_4. To generate our special solutions, we set one free variable equal to 1, and the rest equal to 0.

So, to find our first special solution we set $x_2 = 1$ and $x_4 = 0$. Back substitution (from row 2) gives us $x_3 = 0$, and then (from row 1) $x_1 = -1$. To find our second special solution we set $x_2 = 0$ and $x_4 = 1$. Back substitution now gives $x_3 = -1$ and $x_1 = -1$. So, our two special solutions are the vectors

$$\mathbf{s}_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \qquad\qquad \mathbf{s}_2 = \begin{pmatrix} -1 \\ 0 \\ -1 \\ 1 \end{pmatrix}$$

These two vectors are linearly independent, and the nullspace will be the subspace of $\mathbb{R}^4$ spanned by these vectors

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[1]Death from above!

$$\mathbf{N}(A) = span \left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ -1 \\ 1 \end{pmatrix} \right\}$$

Note something interesting about the columns of A. The first column is exactly the same as the second, while the fourth column is the sum of the first and the third. So, there are only two linearly independent columns, and the dimension of the column space is 2. The nullspace is a subspace of $\mathbb{R}^4$, which is four-dimensional. The column space is two-dimensional. So is the nullspace. And, just in case you didn't know, $2 + 2 = 4$.[2] This is not a coincidence. In general the dimension of the column space, sometimes known as the *rank* of the matrix, plus the dimension of the nullspace, sometimes known as the *nullity* of the matrix, is equal to the dimension of the domain. Said a bit more compactly

$$dom(A) = rank(A) + null(A).$$

More on this later.

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[2]Don't go all 1984 on me here.

Example - Perform elimination on the following matrix, find the pivot and free variables, find the special solutions, and determine the nullspace.

$$A = \begin{pmatrix} -1 & 3 & 5 \\ -2 & 6 & 7 \end{pmatrix}$$

2 Even Simplifieder

At the conclusion of forward elimination we should have a matrix in "stair-case" form. This is called an *echelon matrix*. It will look, in general, something like this

$$U = \begin{pmatrix} p & x & x & x & x & x & x \\ 0 & p & x & x & x & x & x \\ 0 & 0 & 0 & 0 & 0 & p & x \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Each row will have 0 entries until the pivot. The pivots will not be 0. All the entries below the pivots will be 0. The rest of the entries can be anything.[3] For the matrix above there are three pivots, in columns $1, 2$, and 6, and consequently three pivot variables x_1, x_2, x_6. The other variables, x_3, x_4, x_5, x_7, will be free, and as there are four of them, the nullspace will have four special solutions, and will be four-dimensional.

But wait, there's more! We ain't done yet. We don't alter the nullspace if we divide each row by its pivot, and then eliminate all the terms *above* the pivots as well. This will turn our matrix above into something that looks like

$$R = \begin{pmatrix} 1 & 0 & x & x & x & 0 & x \\ 0 & 1 & x & x & x & 0 & x \\ 0 & 0 & 0 & 0 & 0 & 1 & x \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

A matrix in this form is in *reduced row echelon form*, and for matrices in this form figuring out the special solutions is super easy.

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[3]Note that not all the entries with a p are meant to be the same number, nor are all entries with an x.

Example - For the matrix from the last example, write it in reduced row echelon form. Figure out the special solutions by inspection.

$$A = \begin{pmatrix} -1 & 3 & 5 \\ -2 & 6 & 7 \end{pmatrix}$$