Math 2270 - Lecture 13: The Concept of a Nullspace

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This is the first of two lectures covering section 3.2 of the textbook.

In this lecture and the next, we're going to study the space of vectors $\mathbf{x}$ that, for a given matrix A, satisfy the equation $A\mathbf{x} = \mathbf{0}$. This space of vectors is called the *nullspace* of the matrix A, or sometimes the *kernel* of the matrix A.

1 The Idea of the Nullspace

Suppose we have an $m \times n$ matrix A (note that m and n are not necessarily the same, so we're allowing rectangular matrices). We want to know for what vectors $\mathbf{x}$ it is the case that $A\mathbf{x} = \mathbf{0}$. This vector $\mathbf{x}$ will be a vector in $\mathbb{R}^n$, and without knowing anything else about the matrix, sight unseen, I can tell you what one solution will be. One solution is $\mathbf{x} = 0$. The nullspace *always* contains the 0 vector. Sometimes that's it.

Example - What is the nullspace of the matrix

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 8 \\ 3 & 6 & 8 \end{pmatrix}$$

In our last lecture we learned that a subset of a vector space is itself a vector space, called a subspace, if it satisfies two closure axioms. Precisely, we say a subset W of a vector space V is a subspace if:

1. For $\mathbf{w}_1, \mathbf{w}_2 \in W$ we have $\mathbf{w}_1 + \mathbf{w}_2 \in W$.

2. For $\mathbf{w} \in W$ we have $c\mathbf{w} \in W$ for all $c \in \mathbb{R}$.[1]

Well, let's look at the set of all solutions to the equation $A\mathbf{x} = \mathbf{0}$. If $\mathbf{x}_1, \mathbf{x}_2$ are solutions then

$$A(\mathbf{x}_1 + \mathbf{x}_2) = A\mathbf{x}_1 + A\mathbf{x}_2 = \mathbf{0} + \mathbf{0} = \mathbf{0}.$$

Put another way, if $\mathbf{x}_1$ and $\mathbf{x}_2$ are solutions then so is $\mathbf{x}_1 + \mathbf{x}_2$. Similarly, if $\mathbf{x}$ is a solution then

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[1]Here we're assuming our scalar field is $\mathbb{R}$, as we usually do.

$$A(c\mathbf{x}) = cA\mathbf{x} = c\mathbf{0} = \mathbf{0}.$$

So, if $\mathbf{x}$ is a solution then $c\mathbf{x}$ is a solution for any $c \in \mathbb{R}$.

What these two results mean is that the set of all solutions to $A\mathbf{x} = \mathbf{0}$ is a *subspace* of $\mathbb{R}^n$. This is the reason the term "space" appears in the term *nullspace*. It is the space of all solutions to $A\mathbf{x} = \mathbf{0}$, and is denoted $\mathbf{N}(A)$.

Example - What are the nullspaces of the following matrices:

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 6 \\ 3 & 7 & 9 \end{pmatrix} \qquad C = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

The nullspace of the matrix $A = \begin{pmatrix} 2 & 3 & 5 \end{pmatrix}$ is the set of all solutions $A\mathbf{x} = \mathbf{0}$. We can write this as the set of all solutions to the equation

$$x + 3y + 5z = 0.$$

This is the plane through the origin with normal vector $\begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix}$.

2   Bases of Nullspaces

Now, let's take a closer look at the nullspace of the matrix

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix}$$

The nullspace of this matrix will be the set of all solutions $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ to the systems of linear equations

$$\begin{array}{rcrcl} x_1 & + & 2x_2 & = & 0 \\ 3x_1 & + & 6x_2 & = & 0 \end{array}$$

This is really just one equation. Any choice of x_1, x_2 that satisfy the first equation will automatically satisfy the second equation. If we set x_2 arbitrarily, we see $x_1 = -2x_2$, and in general the nullspace of A consists of all multiples of $\mathbf{s} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$.

We call the vector $\mathbf{s} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$ a "special solution" to the system $A\mathbf{x} = \mathbf{0}$, and the nullspace of A consists of all multiples of this "special solution".

We'll be discussing how exactly we find these "special solutions" in the next lecture, but today we'll just do a few more examples.

The plane $x + 3y + 5z$ has *two* special solutions

$$\mathbf{s}_1 = \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix}, \qquad \text{and} \qquad \mathbf{s}_2 = \begin{pmatrix} -5 \\ 0 \\ 1 \end{pmatrix}$$

What is "special" about $\mathbf{s}_1$ and $\mathbf{s}_2$ is that they have ones and zeros in the last two components. Those components are "free" and we choose them "specially". Once we've chosen these free components, the first component is set and forced to equal -3 and -5, respectively. The reason the last

two columns are free, while the first column is set, is that the first column of the matrix $\begin{pmatrix} 1 & 3 & 5 \end{pmatrix}$ contains the *pivot*.

Let's look at some more examples. In particular, let's look at the nullspaces of the three matrices A, B, C:

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 8 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 2 \\ 3 & 8 \\ 1 & 5 \\ 2 & 10 \end{pmatrix} \qquad C = \begin{pmatrix} 1 & 2 & 2 & 4 \\ 3 & 8 & 6 & 16 \end{pmatrix}$$

If we perform elimination on the first matrix A we get

$$\begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix}$$

which has two pivots, one in each column. None of the components are free, and the only solution to $A\mathbf{x} = \mathbf{0}$ is the "trivial" solution $\mathbf{x} = \mathbf{0}$.

The first two rows of the matrix B give us the matrix A. The next two rows just add *more* equations that must be satisfied, and so provide more restrictions. The first two rows already require that the only solution is $\mathbf{x} = \mathbf{0}$, and the two new rows do not, and in fact cannot, change this.[2]

Finally, for the matrix C, if we perform elimination we get the following matrix

$$\begin{pmatrix} 1 & 2 & 2 & 4 \\ 0 & 2 & 0 & 4 \end{pmatrix}$$

Here, the first and second columns are *pivot* columns, while the third and fourth columns are *free* columns.

If we choose $x_3 = 1$ and $x_4 = 0$ for the free variables we get $x_1 = -2$ and $x_2 = 0$. If we choose $x_3 = 0$ and $x_4 = 1$ for the free variables we get $x_1 = 0$ and $x_2 = -2$. So, the two "special" solutions are

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[2]The zero vector is *always* a solution, no matter what, and if it's the only solution then putting more restrictions on our system cannot increase the number of solutions, nor can it decrease it past this one solution.

$$\mathbf{s}_1 = \begin{pmatrix} -2 \\ 0 \\ 1 \\ 0 \end{pmatrix} \qquad \mathbf{s}_2 = \begin{pmatrix} 0 \\ -2 \\ 0 \\ 1 \end{pmatrix}$$

All other solutions to $C\mathbf{x} = \mathbf{0}$ are linear combinations of the above two special solutions.

Now, we don't have to stop elimination after we eliminate downward. We can also eliminate upward and/or divide a whole row by its pivot. These operations don't change the nullspace. When we do this, determining our set components from our free components is even easier. The totally reduced form of C will be

$$R = \begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \end{pmatrix}$$

Using this totally reduced form, it's even easier to solve for x_1 and x_2 given x_3 and x_4.

Example - Completely reduce, and find special solution to, the equation $A\mathbf{x} = \mathbf{0}$ for the matrix

$$A = \begin{pmatrix} 0 & -2 & 3 \\ 4 & 0 & 11 \end{pmatrix}.$$