Math 2270 - Lecture 12: Spaces of Vectors

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This lecture covers **section 3.1** of the textbook.

We now move into a deeper study of the mathematics of linear algebra, beginning with a focus on vector spaces and subspaces. While the connection between vector spaces, subspaces, and solving linear equations might not at first be obvious, the two are deeply and fundamentally connected. We'll be developing this connection over the next few weeks. But first, let's talk about vector spaces.

1 Spaces of Vectors

We'll begin by focusing on the most important vector spaces, the spaces given by column vectors with n components in the real numbers.[1]

Definition - The space $\mathbb{R}^n$ consists of all column vectors $\mathbf{v}$ with n components.

We can represent vectors in $\mathbb{R}^2$ by arrows in the xy-plane, and the xy-plane itself is the vector space $\mathbb{R}^2$. The same idea applies to $\mathbb{R}^3$, although the vectors are harder to draw, or even for $\mathbb{R}^n$ for $n > 3$, although the vectors are *much* harder to draw.

We can add two vectors in a vector space together to get another vector in that vector space, or we can multiply a vector by a constant to get another vector in that vector space. We can combine these two operations to

[1] More generally, instead of restricting ourselves to real numbers, we can extend these definitions to any *field*, but we won't get into that in this class.

construct a linear combination of two vectors, and still get a vector in our vector space. So, if $\mathbf{u}, \mathbf{v}$ are vectors in a vector space, and c, d are scalars, then $c\mathbf{u} + d\mathbf{v}$ is also a vector in the vector space. Note that if $c\mathbf{v}$ is always in our vector space, then we can pick $c = 0$ to get that the zero vector $\mathbf{0}$ is *always* in our vector space.

Example - Why is the set of all polynomials of degree d *not* a vector space[2], while the set of all polynomials of degree *up to* d is?

If we restrict ourselves to, say, degree 2 we can have $(x^2+2x+1) - (x^2+2)$
$= 2x-1$
a polynomial of degree 1.

This is the only issue, and it's cleared up if we allow polynomials of degree <u>up to</u> d.

2 Subspaces

The space $\mathbb{R}^3$, the set of all column vectors with three components, is a vector space. But, what if we restrict ourselves to column vectors where the third components is 0? That is to say, column vectors of the form

[2]The textbook says otherwise, but this is a mistake.

$$\begin{pmatrix} a \\ b \\ 0 \end{pmatrix}$$

We note that if we add any two vectors of this form, we get another vector of this form, and if we multiply any vector of this form by a scalar, we get another vector of this form. In fact, any linear combination of vectors of this form is a vector of this form. So, the set of column vectors of this form is itself in fact a vector space, and is considered a *subspace* of $\mathbb{R}^3$.

Definition - A *subspace* of a vector space is a set of vectors that satisfies two requirements: If $\mathbf{v}$ and $\mathbf{w}$ are vectors in the subspace and c is any scalar, then

(i) $\mathbf{v} + \mathbf{w}$ is in the subspace.

(ii) $c\mathbf{v}$ is in the subspace.

Note first that property (ii) implies that every subspace contains the zero vector. In $\mathbb{R}^3$ some examples of subspaces are $\mathbb{R}^3$ itself, any plane through the origin, any line through the origin, and the zero vector itself. In fact, these are *all* the subspaces of $\mathbb{R}^3$.

Example - In $\mathbb{R}^2$ is the space of all vectors of the form $\begin{pmatrix} a \\ a \end{pmatrix}$ a subspace of $\mathbb{R}^2$? What about the space of all vectors of the form $\begin{pmatrix} a \\ b \end{pmatrix}$ where $a, b \geq 0$?

Vectors of the form $\begin{pmatrix} a \\ a \end{pmatrix}$ are a subspace of $\mathbb{R}^2$.

Vectors of the form $\begin{pmatrix} a \\ b \end{pmatrix}$ where $a, b \geq 0$ are not. We have, for example $-1\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -1 \\ -2 \end{pmatrix}$.

So, $c\vec{v}$ is not in the set when $\vec{v}$ is.

3 The Column Space of A

If we have a matrix A remember that $A\mathbf{x}$ is a combination of the columns of the matrix A, and if we look at all possible vectors $\mathbf{x}$, we get all linear combinations of the columns of A. The set of all linear combinations of the columns of A is a vector space, and it's called the *column space* of A. The question of if $A\mathbf{x} = \mathbf{b}$ has a solution is the question of whether $\mathbf{b}$ is in the column space of A.

Example - Describe the column spaces of these particular matrices:

$$A = \begin{pmatrix} 1 & 2 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 2 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} 1 & 0 \\ 2 & 0 \\ 0 & 0 \end{pmatrix}$$

A - Vectors of the form $\begin{pmatrix} a \\ 0 \\ 0 \end{pmatrix}$ where $a \in \mathbb{R}$. One-dimensional

B - Vectors of the form $\begin{pmatrix} a \\ b \\ 0 \end{pmatrix}$ where $a, b \in \mathbb{R}$. Two-dimensional.

C - Vectors of the form $\begin{pmatrix} a \\ 2a \\ 0 \end{pmatrix}$ where $a \in \mathbb{R}$. One-dimensional.