Math 2270 - Lecture 10: LU Factorization

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Fall 2012

This lecture covers **section 2.6** of the textbook.

1 The Matrices L and U

In elimination what we do is we take a system of equations and convert it into an upper-triangular system. Viewed from the matrix perspective, what we're doing is taking an equation

$$A\mathbf{x} = \mathbf{b}$$

and finding an elimination matrix E such that EA is upper-triangular. The system

$$EA\mathbf{x} = E\mathbf{b}$$

then becomes much easier to solve. If we write $EA = U$, indicating EA is upper-triangular, then our equation is

$$U\mathbf{x} = E\mathbf{b}$$

What might not be obvious here is that the matrix E is lower-triangular and invertible, and on top of that its inverse E^{-1} is also lower-triangular. Denote this inverse $E^{-1} = L$. Then if we multiply both sides of the above equation by $L = E^{-1}$ we get

$$LU\mathbf{x} = LE\mathbf{b} = E^{-1}E\mathbf{b} = \mathbf{b}$$

This looks an awful lot like our original equation $A\mathbf{x} = \mathbf{b}$, and in fact it is our original equation in disguise. This is because $A = LU$. So, we've *factored* A as the product of two matrices, one upper-triangular and the other lower-triangular. Note that throughout this discussion and for the rest of the lecture we'll assume our matrix A is invertible.

2 The Nuts and Bolts of LU Factoriztion

We're now going to take a deeper look at LU factorization, using the LU factorization of

$$A = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}$$

as our running example. To get our matrix U we need to perform elimination on the matrix A, and the first step in elimination here is to subtract $\frac{1}{2}$ the first row from the second. This is achieved with the elimination matrix

$$E_{12} = \begin{pmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

We note that the above elimination matrix is lower-triangular. In fact, all our elimination matrices will be lower-triangular, because in elimination we're always subtracting a higher row from a lower row.[1] Performing our first elimination step we obtain

$$E_{12}A = \begin{pmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 1 & 0 \\ 0 & \frac{3}{2} & 1 \\ 0 & 1 & 2 \end{pmatrix}$$

[1] We're going to assume throughout this lecture that we don't need to permute any rows as part of elimination.

All the terms below our pivot in the first column are now 0, so we move on to the second pivot in row 2, and the second column. We want to subtract $\frac{2}{3}$ the second row from the third. This operation is accomplished by the elimination matrix

$$E_{23} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{2}{3} & 1 \end{pmatrix}$$

The result of the next step in elimination is

$$E_{23}E_{12}A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{2}{3} & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 & 0 \\ 0 & \frac{3}{2} & 1 \\ 0 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 1 & 0 \\ 0 & \frac{3}{2} & 1 \\ 0 & 0 & \frac{4}{3} \end{pmatrix}$$

This is the conclusion of elimination, as our transformed matrix is now upper-triangular. We can multiply E_{23} and E_{12} to get our matrix $E = E_{23}E_{12}$ that transforms A directly into U. This matrix is

$$E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{2}{3} & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ \frac{1}{3} & -\frac{2}{3} & 1 \end{pmatrix}$$

Note the $\frac{1}{3}$ term in the bottom-left. This is because as we do elimination we first subtract $\frac{1}{2}$ of row 1 from row 2, and then subtract $\frac{2}{3}$ of the *modified* row 2 from row 3. This means that in the end we subtract $-\frac{2}{3}$ of the original row 2 from row 3, and add $\frac{1}{3}$ of the original row 1 to row 3. Kind of complicated, isn't it? The amazing thing is that this entanglement doesn't show up when we look at the inverse of E. The inverse of E will be

$$E^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ 0 & \frac{2}{3} & 1 \end{pmatrix}$$

What's going on here? Well, the idea is that the matrix E^{-1} takes U and takes it back to A:

$$E^{-1}U = A$$

Now, we derived U from A through elimination. When we do elimination we might change the rows of A, but once those rows become the final rows we see in U we stop. Once we have a pivot row, it never changes again. In our particular example we have

$$\text{Row 3 of } U = (\text{Row 3 of } A) - \tfrac{2}{3}\,(\text{Row 2 of } U).$$

When computing the third row of U, we subtract multiples of earlier rows of U, *not* rows of A. However, if we want to figure out row 3 of A, then doing some algebra on the above equation we get

$$\text{Row 3 of } A = (\text{Row 3 of } U) + \tfrac{2}{3}\,(\text{Row 2 of } U).$$

The equation for row 3 of A involves just the rows of U, and no other row of A.

Finally, one could argue that, in some sense. the factorization $A = LU$ isn't complete. The lower-triangular matrix will always have 1 terms on the diagonal, while the upper-triangular matrix will not. Sometimes we want to make sure the upper-triangular matrix has 1 terms on the diagonal as well, and so we factor our matrix as

$$A = LDU$$

where D is a diagonal matrix consisting of the pivots, L is lower-triangular with 1 entries on the diagonal, and U is upper-triangular with 1 entries on the diagonal.

Example - Factor the matrix

$$A = \begin{pmatrix} 2 & 1 \\ 6 & 8 \end{pmatrix}$$

into $A = LU$ form and into $A = LDU$ form.

Elimination: Subtract 3x row 1 from row 2

$$\Rightarrow \begin{pmatrix} 2 & 1 \\ 0 & 5 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 \\ -3 & 1 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 6 & 8 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 0 & 5 \end{pmatrix}$$

$\begin{pmatrix} 2 & 1 \\ 0 & 5 \end{pmatrix}$ is upper triangular

$$L = \begin{pmatrix} 1 & 0 \\ -3 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 1 \\ 6 & 8 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 0 & 5 \end{pmatrix} \quad \text{LU form}$$

$$\begin{pmatrix} 2 & 1 \\ 0 & 5 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 5 \end{pmatrix}\begin{pmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 1 \\ 6 & 8 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & 5 \end{pmatrix}\begin{pmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{pmatrix} \quad \text{LDU form}$$