Math 2270 - Final

University of Utah

Fall 2012

Name: Key

This is a 2 hour exam. Please show all your work, as a worked problem is required for full points, and partial credit may be rewarded for some work in the right direction.

1. *Elimination and LU Factorization*

(a) Use elimination to calculate the row echelon form of the matrix

$$A = \begin{pmatrix} 1 & -2 & 3 \\ -1 & 3 & 0 \\ 2 & -5 & 5 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -2 & 3 \\ -1 & 3 & 0 \\ 2 & -5 & 5 \end{pmatrix} \xrightarrow[\text{to row 2}]{\text{Add row 1}} \begin{pmatrix} 1 & -2 & 3 \\ 0 & 1 & 3 \\ 2 & -5 & 5 \end{pmatrix}$$

$$\xrightarrow[\text{from row 3}]{\text{Subtract } 2\times \text{ row 1}} \begin{pmatrix} 1 & -2 & 3 \\ 0 & 1 & 3 \\ 0 & -1 & -1 \end{pmatrix} \xrightarrow[\text{to row 3}]{\text{Add row 2}} \begin{pmatrix} 1 & -2 & 3 \\ 0 & 1 & 3 \\ 0 & 0 & 2 \end{pmatrix}$$

(b) Calculate the LU factorization of the matrix A from part (a) of this problem, given below

$$A = \begin{pmatrix} 1 & -2 & 3 \\ -1 & 3 & 0 \\ 2 & -5 & 5 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -2 & 3 \\ -1 & 3 & 0 \\ 2 & -5 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -2 & 3 \\ 0 & 1 & 3 \\ 0 & 0 & 2 \end{pmatrix}$$

(c) Is the matrix A invertible? How do you know?

Yes. It has no free columns. All the pivots are non-zero.

2. *Subspaces*

(a) Is the set of all vectors (a_1, a_2, a_3) with $a_3 = 1$ a subspace of $\mathbb{R}^3$?

No. It's not closed under addition:

$$\begin{pmatrix} a_1 \\ a_2 \\ 1 \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \\ 1 \end{pmatrix} = \begin{pmatrix} a_1 + b_1 \\ a_2 + b_2 \\ 2 \end{pmatrix}$$

(b) Is the set of all vectors (a_1, a_2, a_3) with $a_3 = 0$ a subspace of $\mathbb{R}^3$?

Yes.

Closed under addition

$$\begin{pmatrix} a_1 \\ a_2 \\ 0 \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \\ 0 \end{pmatrix} = \begin{pmatrix} a_1 + b_1 \\ a_2 + b_2 \\ 0 \end{pmatrix}$$

Closed under multiplication

$$c \begin{pmatrix} a_1 \\ a_2 \\ 0 \end{pmatrix} = \begin{pmatrix} ca_1 \\ ca_2 \\ 0 \end{pmatrix}$$

Third component still 0.

3. *Elminination, Rank, and Complete Solutions*

(a) Use elimination to calculate the row echelon form of the matrix

$$B = \begin{pmatrix} 2 & 4 & 6 & 4 \\ 2 & 5 & 7 & 6 \\ 2 & 3 & 5 & 2 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 4 & 6 & 4 \\ 2 & 5 & 7 & 6 \\ 2 & 3 & 5 & 2 \end{pmatrix} \xrightarrow[\text{1 from row 2}]{\text{Subtract row}} \begin{pmatrix} 2 & 4 & 6 & 4 \\ 0 & 1 & 1 & 2 \\ 2 & 3 & 5 & 2 \end{pmatrix}$$

$$\xrightarrow[\text{row 3}]{\text{Subtract row 1 from}} \begin{pmatrix} 2 & 4 & 6 & 4 \\ 0 & 1 & 1 & 2 \\ 0 & -1 & -1 & -2 \end{pmatrix} \xrightarrow[\text{2 to row 3}]{\text{Add row}} \begin{pmatrix} 2 & 4 & 6 & 4 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

(b) What is the rank of the matrix B from part (a) if this problem?

There are 2 pivots, so the rank is $\boxed{2}$.

(c) What are the dimensions of the following subspaces for the matrix B:

- Row space $\mathbf{C}(B^T)$ - $\cancel{4-2=}\ \boxed{2}$
- Nullspace $\mathbf{N}(B)$ - $4-2=\boxed{2}$
- Column space $\mathbf{C}(B)$ - $\boxed{2}$
- Left nullspace $\mathbf{N}(B^T)$ - $3-2=\boxed{1}$

(d) What is a basis for the nullspace, $\mathbf{N}(B)$, of the matrix B from part (a) of this problem?

$$\begin{pmatrix} 2 & 4 & 6 & 4 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ 1 \\ 0 \end{pmatrix}$$

2 free columns. Choose $x_3 = 1$, $x_4 = 0$

$$\Rightarrow \begin{aligned} 2x_1 + 4x_2 &= -6 \\ x_2 &= -1 \end{aligned} \Rightarrow \begin{aligned} x_1 &= -1 \\ x_2 &= -1 \end{aligned} \quad \begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix}$$

$x_3 = 0$, $x_4 = 1$

$$\begin{pmatrix} 2 & 4 & 6 & 4 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 2 \\ -2 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} 2 \\ -2 \\ 0 \\ 1 \end{pmatrix}$$

So, a basis is:

$$\begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ -2 \\ 0 \\ 1 \end{pmatrix}$$

(e) Find the complete solution to the system of equations

$$\begin{aligned} 2x_1 &+ 4x_2 &+ 6x_3 &+ 4x_4 &= 4 \\ 2x_1 &+ 5x_2 &+ 7x_3 &+ 6x_4 &= 5 \\ 2x_1 &+ 3x_2 &+ 5x_3 &+ 2x_4 &= 3 \end{aligned}$$

$$\begin{pmatrix} 2 & 4 & 6 & 4 & 4 \\ 2 & 5 & 7 & 6 & 5 \\ 2 & 3 & 5 & 2 & 3 \end{pmatrix} \xrightarrow{\text{Subtract row1 from rows 2 and 3}} \begin{pmatrix} 2 & 4 & 6 & 4 & 4 \\ 0 & 1 & 1 & 2 & 1 \\ 0 & -1 & -1 & -2 & -1 \end{pmatrix}$$

$$\xrightarrow{\text{Add row 2 to row 3}} \begin{pmatrix} 2 & 4 & 6 & 4 & 4 \\ 0 & 1 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 2 & 4 & 6 & 4 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \\ 0 \end{pmatrix}$$

Checks Out

Set $x_3 = x_4 = 0$ to get $x_1 = 0$, $x_2 = 1$

So, complete solution is:

$$\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \underbrace{x_3 \begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 2 \\ -2 \\ 0 \\ 1 \end{pmatrix}}_{\text{Nullspace}}$$

4. *Projections*

Calculate the projection of the vector **a** onto the vector **b**:

$$\mathbf{a} = \begin{pmatrix} 1 \\ 2 \\ -3 \\ 1 \end{pmatrix} \qquad \mathbf{b} = \begin{pmatrix} 3 \\ 2 \\ -2 \\ 4 \end{pmatrix}$$

$$\mathrm{proj}_{\vec{b}}(\vec{a}) = \frac{\vec{a} \cdot \vec{b}}{\vec{b} \cdot \vec{b}} \vec{b}$$

$$\vec{a} \cdot \vec{b} = 3 + 4 + 6 + 4 = 17$$

$$\vec{b} \cdot \vec{b} = 9 + 4 + 4 + 16 = 33$$

$$\mathrm{proj}_{\vec{b}}(\vec{a}) = \boxed{\frac{17}{33} \begin{pmatrix} 3 \\ 2 \\ -2 \\ 4 \end{pmatrix}}$$

5. *Eigenvalues and Diagonalization*

(a) Calculate the eigenvalues of the matrix

$$C = \begin{pmatrix} 1 & 0 & 2 \\ 0 & -1 & -2 \\ 2 & -2 & 0 \end{pmatrix}$$

$$\begin{vmatrix} 1-\lambda & 0 & 2 \\ 0 & -1-\lambda & -2 \\ 2 & -2 & -\lambda \end{vmatrix} = (1-\lambda)(-1-\lambda)(-\lambda) - 4(1-\lambda) - 4(-1-\lambda)$$

$$= (1-\lambda)(-1-\lambda)(-\lambda) + 8\lambda$$

$$= (\lambda^2-1)(-\lambda) + 8\lambda = \lambda(9-\lambda^2)$$

$$= \lambda(3-\lambda)(3+\lambda)$$

So, $\lambda = 0, 3, -3$ are the eigenvalues.

(b) Calculate normalized eigenvectors for the matrix C from part (a).

$\lambda = 0$

$$\begin{pmatrix} 1 & 0 & 2 \\ 0 & -1 & -2 \\ 2 & -2 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \vec{x} = \begin{pmatrix} -2 \\ -2 \\ 1 \end{pmatrix} \quad \vec{q}_1 = \begin{pmatrix} -\frac{2}{3} \\ -\frac{2}{3} \\ \frac{1}{3} \end{pmatrix}$$

$\lambda = 3$

$$\begin{pmatrix} -2 & 0 & 2 \\ 0 & -4 & -2 \\ 2 & -2 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \vec{x} = \begin{pmatrix} 1 \\ -\frac{1}{2} \\ 1 \end{pmatrix} \quad \vec{q}_2 = \begin{pmatrix} \frac{2}{3} \\ -\frac{1}{3} \\ \frac{2}{3} \end{pmatrix}$$

$\lambda = -3$

$$\begin{pmatrix} 4 & 0 & 2 \\ 0 & 2 & -2 \\ 2 & -2 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \vec{x} = \begin{pmatrix} -\frac{1}{2} \\ 1 \\ 1 \end{pmatrix} \quad \vec{q}_3 = \begin{pmatrix} -\frac{1}{3} \\ \frac{2}{3} \\ \frac{2}{3} \end{pmatrix}$$

Normalized eigenvectors

$$\begin{pmatrix} -\frac{2}{3} \\ -\frac{2}{3} \\ \frac{1}{3} \end{pmatrix}, \begin{pmatrix} \frac{2}{3} \\ -\frac{1}{3} \\ \frac{2}{3} \end{pmatrix}, \begin{pmatrix} -\frac{1}{3} \\ \frac{2}{3} \\ \frac{2}{3} \end{pmatrix}$$

(c) Diagonalize, $C = S\Lambda S^{-1}$, the matrix C from part (a).

$$\begin{pmatrix} 1 & 0 & 2 \\ 0 & -1 & -2 \\ 2 & -2 & 0 \end{pmatrix} = \boxed{\begin{pmatrix} -\frac{2}{3} & \frac{2}{3} & -\frac{1}{3} \\ -\frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -3 \end{pmatrix} \begin{pmatrix} -\frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \\ \frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \\ -\frac{1}{3} & \frac{2}{3} & \frac{2}{3} \end{pmatrix}}$$

<u>Check:</u>

$$\begin{pmatrix} 0 & 2 & 1 \\ 0 & -1 & -2 \\ 0 & 2 & -2 \end{pmatrix} \begin{pmatrix} -\frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \\ \frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \\ -\frac{1}{3} & \frac{2}{3} & \frac{2}{3} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -2 \\ 2 & -2 & 0 \end{pmatrix}$$

(d) Is the matrix C positive-definite? Why or why not?

No. The eigenvalues are not all positive.

6. *Singular Value Decomposition and the Pseudoinverse*

(a) Calculate the singular value decomposition (SVD) of the matrix

$$D = \begin{pmatrix} 2 & 2 \\ 1 & 1 \end{pmatrix}$$

$\cancel{2-\lambda}$

$$D^T D = \begin{pmatrix} 2 & 1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 2 & 2 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 5 & 5 \\ 5 & 5 \end{pmatrix}$$

$$\begin{vmatrix} 5-\lambda & 5 \\ 5 & 5-\lambda \end{vmatrix} = (5-\lambda)^2 - 25 = \lambda^2 - 10\lambda = \lambda(\lambda - 10)$$

$\lambda = 10, 0$ Singular value $\sigma_1 = \sqrt{10}$

$$\begin{pmatrix} -5 & 5 \\ 5 & -5 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \vec{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \vec{v}_1 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$D\vec{v}_1 = \begin{pmatrix} 2 & 2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} 2\sqrt{2} \\ \sqrt{2} \end{pmatrix} \quad \vec{u}_1 = \frac{1}{\sqrt{10}}\begin{pmatrix} 2\sqrt{2} \\ \sqrt{2} \end{pmatrix} = \begin{pmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{pmatrix}$$

$$\boxed{\begin{pmatrix} 2 & 2 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{pmatrix} \left(\sqrt{10}\right) \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}}$$

$D = U \Sigma V^T$

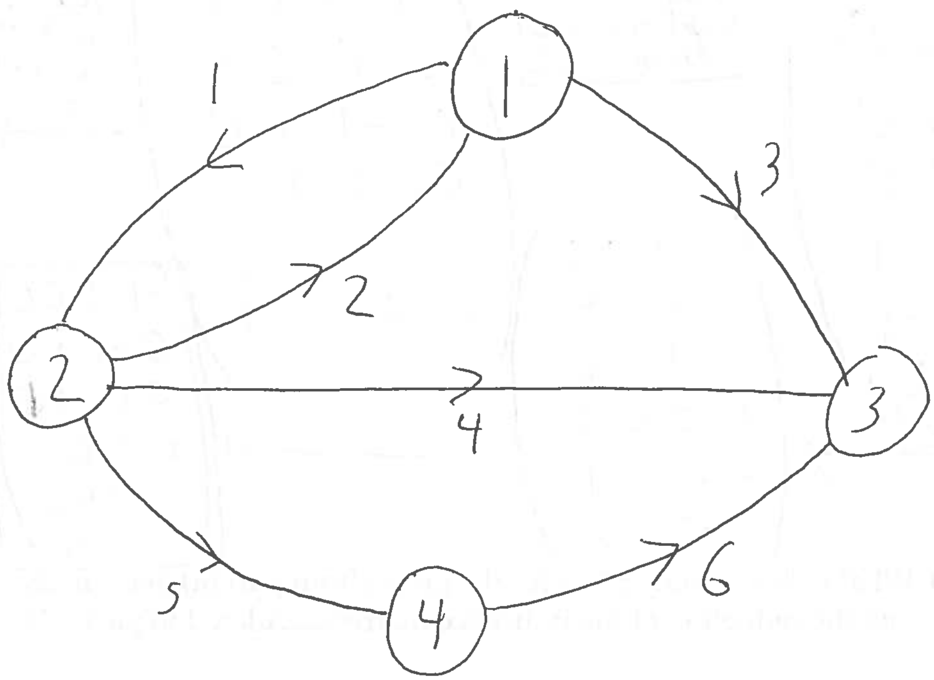
(b) Calculate the pseudoinverse of the matrix D from part (a).

$$\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \left(\frac{1}{\sqrt{10}} \right) \begin{pmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix} = \boxed{\begin{pmatrix} \frac{1}{5} & \frac{1}{10} \\ \frac{1}{5} & \frac{1}{10} \end{pmatrix}}$$

$$D^+ = V \Sigma^+ U^T$$

7. *Graphs and Incidence Matrices*

(a) What is the incidence matrix for the graph:



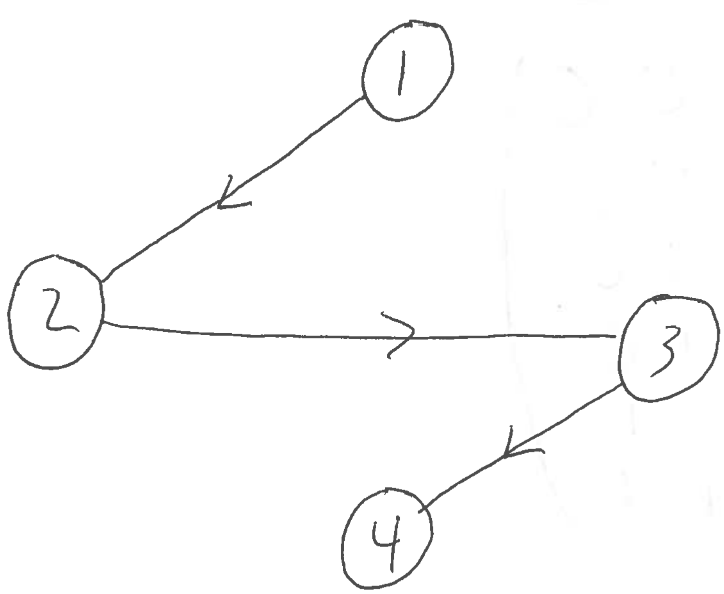
$$\begin{pmatrix} -1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

(b) Perform elimination on the indicence matrix from part (a).

$$\begin{pmatrix} -1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix} \xrightarrow[\text{from row 3}]{\substack{\text{Add row 1 to row 2} \\ \text{Subtract row 1}}} \begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix} \xrightarrow[\text{rows 4 and 5}]{\text{Subtract row 3 from}} \begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

$$\xrightarrow{\text{Add row 5 to row 6}} \begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \longrightarrow \boxed{\begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}}$$

(c) What is the spanning tree for the graph from part (a) determined by the reduction of the indicence matrix calculated in part (b)?



8. *Markov Matrices*

(a) What are the eigenvalues of the Markov matrix

$$M = \begin{pmatrix} .5 & .7 \\ .5 & .3 \end{pmatrix}$$

Hint - It's not necessary to find the roots of the characteristic equation! Just use what you know about eigenvalues of positive Markov matrices, and how the sum of the eigenvalues relates to the trace of the matrix.

$\lambda_1 + \lambda_2 = .5 + .3 = .8$

$\lambda_1 = 1$ as A is a positive Markov matrix.

$\lambda_2 = .8 - 1 = -.2$

$$\boxed{\begin{aligned} \lambda_1 &= 1 \\ \lambda_2 &= -.2 \end{aligned}}$$

(b) What is the steady-state solution for the Markov matrix M from part (a)?

The eigenvector

$$\begin{pmatrix} -.5 & .7 \\ .5 & -.7 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \vec{x} = \begin{pmatrix} .7 \\ .5 \end{pmatrix}$$

$$\vec{u}_\infty = \begin{pmatrix} \frac{.7}{1.2} \\ \frac{.5}{1.2} \end{pmatrix} = \boxed{\begin{pmatrix} \frac{7}{12} \\ \frac{5}{12} \end{pmatrix}}$$