Math 2270 - Practice Final

University of Utah

Fall 2012

Name: Key

This is a 2 hour exam. Please show all your work, as a worked problem is required for full points, and partial credit may be rewarded for some work in the right direction.

1. *Elimination and LU Factorization*

(a) Use elimination to calculate the row-echelon form of the matrix

$$\begin{pmatrix} 1 & 2 & 0 \\ 3 & 7 & 1 \\ 2 & 5 & 2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 0 \\ 3 & 7 & 1 \\ 2 & 5 & 2 \end{pmatrix} \xrightarrow[\text{from row 2}]{\text{Subtract } 3\times \text{ row 1}} \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 2 & 5 & 2 \end{pmatrix}$$

$$\xrightarrow[\text{from row 3}]{\text{Subtract } 2\times \text{ row 1}} \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix} \xrightarrow[\text{row 3}]{\text{Subtract row 2 from}} \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

(b) (15 points) Calculate the LU factorization of the matrix from part (a)

$$\begin{pmatrix} 1 & 2 & 0 \\ 3 & 7 & 1 \\ 2 & 5 & 2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 0 \\ 3 & 7 & 1 \\ 2 & 5 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

(c) (5 points) Is the matrix A positive-definite? How do you know?

No. A positive-definite matrix must be symmetric, and A is not.

2. **(a)** What is the rank of the matrix?

$$\begin{pmatrix} 2 & 3 & 1 & 4 \\ 2 & 4 & 3 & 5 \\ 2 & 5 & 5 & 6 \\ 0 & 3 & 6 & 3 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 3 & 1 & 4 \\ 2 & 4 & 3 & 5 \\ 2 & 5 & 5 & 6 \\ 0 & 3 & 6 & 3 \end{pmatrix} \xrightarrow{\text{Subtract row 1 from row 2}} \begin{pmatrix} 2 & 3 & 1 & 4 \\ 0 & 1 & 2 & 1 \\ 2 & 5 & 5 & 6 \\ 0 & 3 & 6 & 3 \end{pmatrix}$$

$$\xrightarrow{\text{Subtract row 1 from row 3}} \begin{pmatrix} 2 & 3 & 1 & 4 \\ 0 & 1 & 2 & 1 \\ 0 & 2 & 4 & 2 \\ 0 & 3 & 6 & 3 \end{pmatrix} \xrightarrow{\text{Subtract 2x row 2 from row 3}} \begin{pmatrix} 2 & 3 & 1 & 4 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 3 & 6 & 3 \end{pmatrix}$$

$$\xrightarrow{\text{Subtract 3x row 2 from row 4}} \begin{pmatrix} 2 & 3 & 1 & 4 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

2 pivots

$\Rightarrow \boxed{\text{rank } 2}$

(b) What is the nullspace of the matrix? (Express it as the span of a set of linearly independent vectors.)

$$\begin{pmatrix} 2 & 3 & 1 & 4 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{5}{2} \\ -2 \\ 1 \\ 0 \end{pmatrix}$$

1, 2 pivots
3, 4 free

$$\begin{pmatrix} 2 & 3 & 1 & 4 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} -\frac{1}{2} \\ -1 \\ 0 \\ 1 \end{pmatrix}$$

$$\text{Nullspace} = \text{span} \left\{ \begin{pmatrix} \frac{5}{2} \\ -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -\frac{1}{2} \\ -1 \\ 0 \\ 1 \end{pmatrix} \right\}$$

3. Calculate the determinant of the matrix

$$\begin{pmatrix} 2 & 3 & 5 \\ 1 & 2 & 1 \\ 3 & 1 & 4 \end{pmatrix}$$

Is the matrix invertible? Why or why not?

$2 \times 2 \times 4 + 3 \times 1 \times 3 + 5 \times 1 \times 1 - 2 \times 1 \times 1 - 3 \times 1 \times 4 - 5 \times 2 \times 3$

$= 16 + 9 + 5 - 2 - 12 - 30$

$= \boxed{-14}$

Matrix is invertible as its determinant is not 0.

4. Use the Gram-Schmidt process to find an orthonormal basis for the vector space spanned by the vectors

$$\begin{pmatrix} 4 \\ -3 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 4 \end{pmatrix}.$$

$$A = \begin{pmatrix} 4 \\ -3 \\ 0 \end{pmatrix}$$

$$B = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} - \frac{(4\ -3\ 0)\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}}{(4\ -3\ 0)\begin{pmatrix} 4 \\ -3 \\ 0 \end{pmatrix}} \begin{pmatrix} 4 \\ -3 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} - \frac{-2}{25}\begin{pmatrix} 4 \\ -3 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{33}{25} \\ \frac{44}{25} \\ 0 \end{pmatrix} = \frac{11}{25}\begin{pmatrix} 3 \\ 4 \\ 0 \end{pmatrix}$$

$$C = \begin{pmatrix} 0 \\ 0 \\ 4 \end{pmatrix}. \quad A \cdot C = B \cdot C = 0.$$

Orthonormalize: $\boxed{\begin{pmatrix} \frac{4}{5} \\ -\frac{3}{5} \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{3}{5} \\ \frac{4}{5} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}$

5. **(a)** Calculate the eigenvalues of the matrix

$$\begin{pmatrix} 1 & -1 & 2 \\ -1 & 1 & 2 \\ 2 & 2 & 2 \end{pmatrix}$$

$$\begin{vmatrix} 1-\lambda & -1 & 2 \\ -1 & 1-\lambda & 2 \\ 2 & 2 & 2-\lambda \end{vmatrix} = (1-\lambda)(1-\lambda)(2-\lambda) - 4 - 4$$
$$- 4(1-\lambda) - (2-\lambda) - 4(1-\lambda)$$

$$= (1-\lambda)(2-\lambda) - \lambda(1-\lambda)(2-\lambda) - 8 - 4 + 4\lambda$$
$$-2 + \lambda - 4 + 4\lambda$$

$$= 2 - 3\lambda + \lambda^2 - 2\lambda + 3\lambda^2 - \lambda^3 - 12 + 4\lambda - 6 + 5\lambda$$

$$= -\lambda^3 + 4\lambda^2 + 4\lambda - 16$$

$$= (4-\lambda)(\lambda^2 - 4) = (4-\lambda)(\lambda-2)(\lambda+2)$$

$$\lambda = 4, 2, -2$$

(b) Find the diagonalization, $A = S\Lambda S^{-1}$, of the matrix from part (a).

$$\begin{pmatrix} 1 & -1 & 2 \\ -1 & 1 & 2 \\ 2 & 2 & 2 \end{pmatrix}$$

$\lambda = 4$

$$\begin{pmatrix} -3 & -1 & 2 \\ -1 & -3 & 2 \\ 2 & 2 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \to \begin{pmatrix} \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \end{pmatrix}$$

$\lambda = 2$

$$\begin{pmatrix} -1 & -1 & 2 \\ -1 & -1 & 2 \\ 2 & 2 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \to \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}$$

$\lambda = -2$

$$\begin{pmatrix} 3 & -1 & 2 \\ -1 & 3 & 2 \\ 2 & 2 & 4 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \to \begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \end{pmatrix}$$

$$\begin{pmatrix} 1 & -1 & 2 \\ -1 & 1 & 2 \\ 2 & 2 & 2 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ \frac{2}{\sqrt{6}} & 0 & -\frac{1}{\sqrt{3}} \end{pmatrix}\begin{pmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \end{pmatrix}$$

6. What are the possible Jordan canonical forms for a 5×5 matrix with eigenvalues $\lambda = 1, 1, 1, 2, 2$.

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix},$$

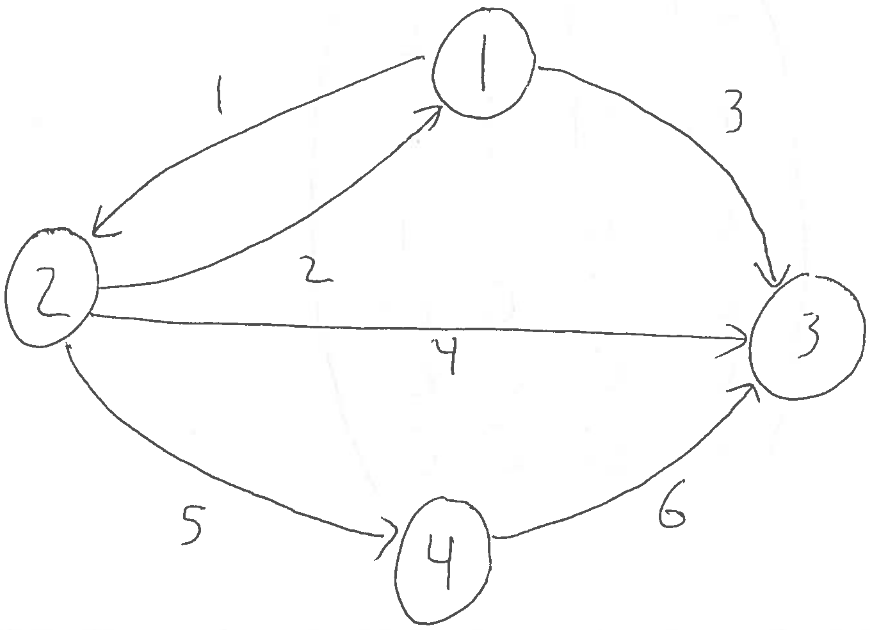
$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix}$$

7. Is the transformation $T(v_1, v_2) = v_1v_2 + v_1 + v_2$ a linear transformation? Why or why not?

No. $T(1,0) = 1 = T(0,1)$

But $T(1,1) = 3 \neq T(1,0) + T(0,1)$

8. **(a)** What is the incidence matrix for the graph:



$$\begin{pmatrix} -1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

(b) Perform elimination on the indicence matrix from part (a).

$$\begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

(c) What is the spanning tree for the graph from part (a) determined by the reduction of the indicence matrix calculated in part (b)?