Math 2270 - Exam 4

University of Utah

Fall 2012

Name: Key

This is a 50 minute exam. Please show all your work, as a worked problem is required for full points, and partial credit may be rewarded for some work in the right direction.

1. *Cofactor Matrices* (20 points)

   Calculate the cofactor matrix of A:

$$A = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ -6 & 2 & 3 \end{pmatrix}$$

$M_{11} = 2 \quad M_{12} = -3 \quad M_{13} = 2$

$M_{21} = -3 \quad M_{22} = 3 \quad M_{23} = -4$

$M_{31} = 1 \quad M_{32} = -1 \quad M_{33} = 1$

$$C = \begin{pmatrix} 2 & 3 & 2 \\ 3 & 3 & 4 \\ 1 & 1 & 1 \end{pmatrix}$$

2. *Eigenvalues* (20 points)

Find the eigenvalues and the corresponding eigenvectors of

$$A = \begin{pmatrix} 2 & 3 & 1 \\ 0 & -1 & 2 \\ 0 & 0 & 3 \end{pmatrix}$$

$$\begin{vmatrix} 2-\lambda & 3 & 1 \\ 0 & -1-\lambda & 2 \\ 0 & 0 & 3-\lambda \end{vmatrix} = (3-\lambda)(2-\lambda)(-1-\lambda)$$

$$\lambda = 3, 2, -1$$

$$\lambda = 3 \quad \begin{pmatrix} -1 & 3 & 1 \\ 0 & -4 & 2 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \vec{x} = \begin{pmatrix} 5 \\ 1 \\ 2 \end{pmatrix}$$

$$\lambda = 2 \quad \begin{pmatrix} 0 & 3 & 1 \\ 0 & -3 & 2 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \vec{x} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\lambda = -1 \quad \begin{pmatrix} 3 & 3 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \vec{x} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

3. *Diagonalization* (20 points)

Diagonalize the matrix

$$A = \begin{pmatrix} 1 & 3 \\ -1 & 5 \end{pmatrix}$$

$$\begin{vmatrix} 1-\lambda & 3 \\ -1 & 5-\lambda \end{vmatrix} = (1-\lambda)(5-\lambda)+3$$
$$= \lambda^2 - 6\lambda + 8 = (\lambda-4)(\lambda-2)$$

$\lambda = 4, 2$

$$\begin{pmatrix} -3 & 3 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \vec{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} -1 & 3 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \vec{x} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

$$S = \begin{pmatrix} 1 & 3 \\ 1 & 1 \end{pmatrix} \quad \det(S) = -2 \quad S^{-1} = \frac{1}{-2}\begin{pmatrix} 1 & -3 \\ -1 & 1 \end{pmatrix}$$

$$\boxed{\begin{pmatrix} 1 & 3 \\ -1 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} -\frac{1}{2} & \frac{3}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}}$$

4. *Positive Definite Matrices* (10 points)

Prove that if R is a matrix with independent columns then $R^T R$ is positive definite. (Hint - A matrix is positive definite if $\mathbf{x}^T A\mathbf{x} > 0$ for all $\mathbf{x} \neq \mathbf{0}$.)

$$\vec{x}^T R^T R \vec{x} = (R\vec{x})^T (R\vec{x})$$
$$= \|R\vec{x}\|^2 > 0$$

As the columns of R are independent $R\vec{x} \neq \vec{0}$, so $\|R\vec{x}\|^2 > 0$.

So, $\vec{x}^T R^T R \vec{x} > 0$ for all $\vec{x}$, and so $R^T R$ is positive definite.

5. *Jordan Form* (10 points)

If a matrix has eigenvalues $\lambda = 2, 2, 2, 1$ what are all the possible Jordan forms of the matrix?

$$\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

6. *Singular Value Decomposition* (20 points)

Calculate the singular value decomposition of the matrix

$$A = \begin{pmatrix} 2 & 2 \\ 1 & 1 \end{pmatrix}$$

$$A^T = \begin{pmatrix} 2 & 1 \\ 2 & 1 \end{pmatrix} \quad A^TA = \begin{pmatrix} 2 & 1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 2 & 2 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 5 & 5 \\ 5 & 5 \end{pmatrix}$$

$$\begin{vmatrix} 5-\lambda & 5 \\ 5 & 5-\lambda \end{vmatrix} = \lambda^2 - 10\lambda \quad \lambda = 10, 0$$

$$\begin{pmatrix} -5 & 5 \\ 5 & -5 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \vec{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \vec{v}_1 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$A\vec{v}_1 = \begin{pmatrix} 2\sqrt{2} \\ \sqrt{2} \end{pmatrix} \quad \vec{u}_1 = \frac{A\vec{v}_1}{\sigma_1} = \frac{1}{\sqrt{10}}\begin{pmatrix} 2\sqrt{2} \\ \sqrt{2} \end{pmatrix} = \begin{pmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{pmatrix}$$

$$\boxed{\begin{pmatrix} 2 & 2 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{pmatrix} \begin{pmatrix} \sqrt{10} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}}$$