Math 2270 - Practice Exam 4

University of Utah

Fall 2012

Name: Key

This is a 50 minute exam. Please show all your work, as a worked problem is required for full points, and partial credit may be rewarded for some work in the right direction.

1. *Cofactor Matrices* (15 points)

Calculate the cofactor matrix of A:

$$A = \begin{pmatrix} 4 & 3 \\ 2 & 8 \end{pmatrix}$$

$$C = \begin{pmatrix} 8 & -2 \\ -3 & 4 \end{pmatrix}$$

2. *Eigenvalues* (20 points)

Find the eigenvalues and the corresponding eigenvectors of

$$A = \begin{pmatrix} 2 & -12 \\ 1 & -5 \end{pmatrix}$$

$$\begin{vmatrix} 2-\lambda & -12 \\ 1 & -5-\lambda \end{vmatrix} = (2-\lambda)(-5-\lambda) - (-12)(1)$$

$$= \lambda^2 + 3\lambda + 2 = (\lambda+2)(\lambda+1)$$

So, eigenvalues $\lambda = -1, -2$

$\lambda = -1$ $\begin{pmatrix} 3 & -12 \\ 1 & -4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $\vec{x} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$

$\lambda = -2$ $\begin{pmatrix} 4 & -12 \\ 1 & -3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $\vec{x} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

Eigenvectors $\begin{pmatrix} 4 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

3. *Diagonalization* (20 points)

Diagonalize the matrix

$$A = \begin{pmatrix} 1 & 3 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

$$\begin{vmatrix} 1-\lambda & 3 & 0 \\ 3 & 1-\lambda & 0 \\ 0 & 0 & -2-\lambda \end{vmatrix} = (-2-\lambda)\left[(1-\lambda)^2 - 9\right]$$

$$= (-2-\lambda)\left[\cancel{x}^2 - 2\lambda - 8\right]$$

$$= -(\lambda+2)(\lambda-4)(\lambda+2)$$

$$\lambda = 4, -2, -2$$

$$\lambda = 4 \quad \begin{pmatrix} -3 & 3 & 0 \\ 3 & -3 & 0 \\ 0 & 0 & -6 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \vec{x} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

$$\lambda = -2 \quad \begin{pmatrix} 3 & 3 & 0 \\ 3 & 3 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \vec{x} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \text{ or } \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

orthonormalize $\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$$A = \begin{pmatrix} 1 & 3 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 4 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

4. *Positive Definite Matrices* (10 points)

Prove that if A is positive definite, and B is positive definite, then $A + B$ is positive definite. (Hint - A matrix is positive definite if $\mathbf{x}^T A\mathbf{x} > 0$ for all $\mathbf{x} \neq 0$.)

Suppose A and B are positive definite. Then

$$\vec{x}^T(A+B)\vec{x} = \vec{x}^T A\vec{x} + \vec{x}^T B\vec{x} > 0$$

for all $\vec{x}$ as $\vec{x}^T A\vec{x}$, $\vec{x}^T B\vec{x} > 0$ for all $\vec{x}$.

5. *Jordan Form*

If a matrix has eigenvalues $\lambda = 2, 2, 1, 0$ what are all the possible Jordan forms of the matrix?

$$\begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

6. *Singular Value Decomposition* (20 points)

Calculate the singular value decomposition of the matrix

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$

$$A^T A = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

$$\begin{vmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 - 1 = \lambda^2 - 2\lambda = \lambda(\lambda - 2)$$

$$\lambda = 2, 0$$

$$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \vec{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \vec{v}_1 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$A\vec{v}_1 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \sqrt{2} \\ 0 \end{pmatrix} = \sqrt{2}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \vec{u}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\sigma_1 = \sqrt{2}$$

$$A = U\Sigma V^T = \boxed{\begin{pmatrix} 1 \\ 0 \end{pmatrix} (\sqrt{2}) \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}} \leftarrow \text{SVD}$$

$$\begin{pmatrix} \sqrt{2} \\ 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \checkmark$$

7. *Linear Transformations* (15 points)

(a) Is the transformation $T : \mathbb{R}^2 \to \mathbb{R}^1$ defined by

$$T(v_1, v_2) = 2v_1 + v_2$$

a linear transformation? Explain why or provide a counter-example.

$$T(u_1+v_1, u_2+v_2) = 2(u_1+v_1) + u_2+v_2$$

$$= (2u_1+u_2) + (2v_1+v_2) = T(u_1,u_2) + T(v_1,v_2)$$

$$T(cv_1, cv_2) = 2cv_1 + cv_2 = c(2v_1+v_2)$$

$$= cT(v_1,v_2). \quad \text{So, } \boxed{\text{linear}}$$

(b) Is the transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$ defined by

$$T(v_1, v_2) \to (2v_1, 3v_1 + 2v_2, v_1v_2)$$

a linear transformation? Explain why or provide a counter-example.

$$\vec{u} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \vec{v} = \begin{pmatrix} 0 \\ \cancel{0}1 \end{pmatrix}$$

$$T(\vec{u}+\vec{v}) = T(1,1) = (2,5,1)$$

$$\times$$

$$T(u) + T(v) = (2,3,0) + (0,2,0) = (2,5,0)$$

$$\text{So, } \boxed{\text{not linear}}.$$